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Brief paper

Robust data-driven Kalman filtering for unknown linear systems using maximum likelihood optimization☆

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ABSTRACT

This paper investigates the state estimation problem for unknown linear systems subject to both process and measurement noise. Based on a prior input–output trajectory sampled at a higher frequency and a prior state trajectory sampled at a lower frequency, we propose a novel robust data-driven Kalman filter (RDKF) that integrates model identification with state estimation for the unknown system. Specifically, the state estimation problem is formulated as a non-convex maximum likelihood optimization problem. Then, we slightly modify the optimization problem to get a problem solvable with a recursive algorithm. Based on the optimal solution to this new problem, the RDKF is designed, which can estimate the state of a given but unknown state-space model. The performance gap between the RDKF and the optimal Kalman filter based on known system matrices is quantified through a sample complexity bound. In particular, when the number of the pre-collected states tends to infinity, this gap converges to zero. Finally, the effectiveness of the theoretical results is illustrated by numerical simulations.

1. Introduction

Due to its ability to estimate states of dynamic systems, Kalman filtering has attracted tremendous attention since its inception in 1960, which has been widely applied in practice (Auger et al., 2013; Kalman, 1960). However, the effectiveness of Kalman filters is dependent on prior knowledge of system dynamics that may be unavailable in some practical implementations (Anderson & Moore, 2005; Netto & Mili, 2018). To tackle this issue, some research efforts have been devoted to learning filters from pre-collected system trajecotries, as described below.

Learning state estimators for unknown systems from pre-collected system data has been a longstanding topic in the control society (Hou & Wang, 2013; Markovsky & Dörfler, 2021; Tsiamis, Matni, & Pappas, 2020; Xian, Zhao, Wen, & Chen, 2024). According to designing criteria, there are two paradigms of data-driven state estimation: *indirect data-driven state estimation*, also referred to as system identification-based state estimation (Alanwar, Berndt, Johansson, & Sandberg, 2022; Revach et al., 2022; Tsiamis et al., 2020), and *direct data-driven state estimation* (Liu, Wang, Sun, Bullo, & Chen, 2024; Mehra, 1970; Shafieezadeh Abadeh, Nguyen, Kuhn, & Mohajerin Esfahani, 2018; Wolff, Lopez, & Müller, 2024). Indirect data-driven state estimation identifies a state-space model using the pre-collected system data following classical system identification approaches (Ljung, 1999), and then designs a state estimator based on the identified model. State-space linear models (Tsiamis et al., 2020; Xian et al., 2024) and neural networks (NNs) (Revach et al., 2022) are commonly adopted. For example, Tsiamis et al. (2020) adopted a subspace identification approach to identify a state-space linear model and later proposed both a certainty equivalent and a robust Kalman filter. Xian et al. (2024) proposed a new model-based external disturbance observer to estimate the unknown time-varying external disturbance, then the optimal coordination problem was solved using a novel and integrated algorithm architecture. Revach et al. (2022) trained an NN model to describe a system before designing a Kalman filter. Direct data-driven state estimation designs a state estimator directly from system data with no intermediate system identification step. This method

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is based on different principles, such as adaptive control theory (Mehra, 1970; Shafieezadeh Abadeh et al., 2018) and Willems' fundamental lemma (Liu et al., 2024; Wolff et al., 2024). The pioneering idea of direct data-driven state estimation can be traced back to adaptive filtering for cases with unknown noise covariances (Mehra, 1970; Shafieezadeh Abadeh et al., 2018). Recently, Willems' fundamental lemma (Willems, Rapisarda, Markovsky, & De Moor, 2005, Theorem 1), which provides a sufficient condition under which an input–output trajectory of a linear time-invariant system can be recovered by a measured input–output trajectory, contributes to several essential results on direct data-driven filtering (Liu et al., 2024; Wolff et al., 2024). For example, Liu et al. (2024) and Wolff et al. (2024) leveraged this lemma to design an explicit observer and an implicit moving horizon estimator for a state-space linear system with unknown system matrices, respectively.

The aforementioned data-driven state estimation methods rely on a prior input-state-output trajectory, where the input, state, and output should be sampled at the same frequency. However, in many practical scenarios, the state sampling frequency is often much lower than that of the input–output trajectory, as measuring the state may require additional sampling strategies and extended sampling times, as illustrated in the motivating example in Section 2. In this case, an unresolved issue is how to leverage a prior input–output trajectory and a lower-frequency sampled state trajectory for online state estimation of an unknown system. Moreover, the impact of data noise on the filtering performance is not yet fully understood in the literature. For example, the specific relationship between the magnitude of data noise and the filtering performance has not been established. Altogether, designing a filtering method that can estimate the state of a given state-space model with unknown system matrices has not well addressed, let along conducting an in-depth analysis of the filtering performance.

Motivated by the above findings, this paper investigates the state estimation problem for a linear system with a pre-defined state but unknown system matrices of the corresponding state-space model. For this system, we pre-collect an input–output trajectory at a higher frequency and a state trajectory at a lower frequency. This paper formulates the modeling and filtering problem for the unknown linear system as a unified maximum likelihood (ML) optimization problem, the solution to which generates a novel robust data-driven Kalman filter (RDKF). In comparison to the literature, this paper possesses several special features as follows:

(1) The RDKF is developed based on a prior input–output trajectory and a lower-frequency sampled state trajectory, providing an alternative for state estimation in practical scenarios where sampling the state at the same frequency as the input–output trajectory is impractical.
(2) A feasibility condition for the RDKF is established, demonstrating that the RDKF is feasible when the prior system trajectory is sufficiently long, and providing a specific requirement on the trajectory length (Theorem 1).
(3) The filtering performance of the RDKF is ensured. Particularly, a sample-complexity bound is derived for the performance gap between the RDKF and the Kalman filter based on known system matrices, which also quantitatively reveals the impact of data noise on the performance (Theorems 2 and 3).
(4) The RDKF is further generalized for cases with only a prior input–output trajectory. In this case, the state estimate corresponds to a balanced realization of the system, which can be applied to control tasks such as LQG control (Corollary 2).

The remainder of this paper is organized as follows. Section 2 presents the problem formulation. Section 3 introduces a novel RDKF algorithm. Section 4 analyzes the necessary informativity of the pre-collected data required for performing the RDKF. Section 5 evaluates the filtering performance of the RDKF. The theoretical results are illustrated in Section 6. Section 7 concludes the paper.

Notations. Let $\mathbb{R}^n$ denote the real coordinate space of dimension n. Let $\otimes$ denote the Kronecker product. Let I_n denote the n-order identity matrix. Let 0 denote a scalar, vector, or matrix of an appropriate dimension with all elements being zero. Let $\mathbb{N}^+$ be the set of positive integers and $\mathbb{N} = \mathbb{N}^+ \cup 0$. For any given vector μ and positive definite matrix Σ with appropriate dimensions, let $\mathcal{N}(\mu, \Sigma)$ denote Gaussian distribution with mean μ and covariance Σ. For any positive function f, let $\ln f$ denote its natural logarithm. For any matrix S, $S(m_1 : m_2; n_1 : n_2)$ denotes the block matrix in S with elements S_{ij}, $m_1 \le i \le m_2$, $n_1 \le j \le n_2$; $S^\dagger$ denotes its right/left inverse if it has full row/column rank. For a square matrix S, $|S|$ denotes its determinant; $\lambda(S)$ denotes the set of its eigenvalues; $\lambda_{\max}(S)/\lambda_{\min}(S)$ denotes its maximum/minimum eigenvalue if S is positive definite.

2. Problem formulation

2.1. System model

This paper considers a class of linear systems:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \omega_k, \\ y_k &= Cx_k + \nu_k, \quad k \in \mathbb{N}, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $y_k \in \mathbb{R}^p$ denote the system state, input, and output at time step k, respectively; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are unknown system state, input, and output matrices, respectively; $\omega_k \in \mathbb{R}^n \sim \mathcal{N}(0, Q)$ and $\nu_k \in \mathbb{R}^p \sim \mathcal{N}(0, R)$ are the system process and measurement noise with $Q \in \mathbb{R}^{n\times n} > 0$ and $R \in \mathbb{R}^{p\times p} > 0$, respectively. The initial system state is denoted by $x_0 \sim \mathcal{N}(\bar{x}_0, P_0)$ with $\bar{x}_0 \in \mathbb{R}^n$ and $P_0 \in \mathbb{R}^{n\times n} > 0$. We assume that x_0, ω_k, and ν_k, $\forall k \in \mathbb{N}$, are mutually uncorrelated.

Assumption 1. (C, A) is observable.

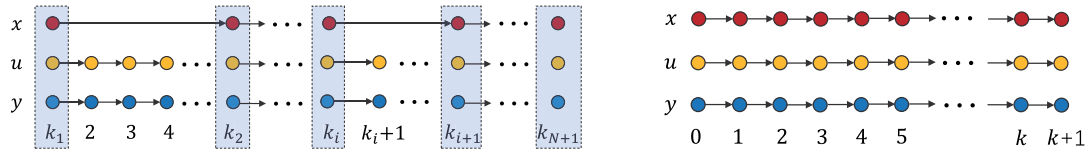
2.2. Data collection

Suppose that we can pre-collect an input-state-output trajectory of (1), as shown in Fig. 1(a) , where the state sampling frequency is lower than the input and output sampling frequency. To be more specific, the sampled state sequence is denoted by

$$x^{\mathrm{p}} = [x_{k_1}^T,\ x_{k_2}^T,\ x_{k_3}^T,\ \ldots,\ x_{k_{N+1}}^T]^T, \tag{2}$$

where the superscript 'p' denotes the pre-collected data. Then, let $L_i = k_{i+1} - k_i$, $i \in \mathcal{V} \triangleq \{1, 2, \ldots, N\}$. Without loss of generality, we assume that there exists a constant scalar L such that $L_i \ge L$, $\forall i \in \mathcal{V}$. If $L_i < L$ for some $i \in \mathcal{V}$, we can omit the sampled state $x_{k_{i+1}}$ and retain the next sampled state until the condition $L_i \ge L$ is satisfied. This assumption is practical, particularly when the state sampling frequency is much lower than the input–output sampling frequency. Next, we divided the input–output trajectory into N segments and then extracted the first L points from each segment, as

$$\begin{aligned} u^{i,\mathrm{p}} &= [u_{k_i}^T,\ u_{k_i+1}^T,\ \ldots,\ u_{k_i+L-1}^T]^T, \\ y^{i,\mathrm{p}} &= [y_{k_i}^T,\ y_{k_i+1}^T,\ \ldots,\ y_{k_i+L}^T]^T, \end{aligned}$$



(a) The pre-collected input-state-output trajectory. (b) The online input-state-output trajectory.

Fig. 1. The pre-collected and online system trajectories, where red, yellow, and blue solid dots denote the state, input, and output, respectively. In Fig. (a), the states are sampled at time instants k_1, k_2, …, k_{N+1}, and the inputs and outputs are sampled at every time instant from k_1 to k_{N+1}. This paper aims to estimate the online state at every time instant, using the pre-collected input-state-output trajectory and the online input–output trajectory. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $i \in \mathcal{V}$ denotes the ith segment. The augmented input and output data of all segments are denoted by

$$\begin{aligned} u^{\mathrm{p}} &= [(u^{1,\mathrm{p}})^T,\ \ldots,\ (u^{N,\mathrm{p}})^T]^T, \\ y^{\mathrm{p}} &= [(y^{1,\mathrm{p}})^T,\ \ldots,\ (y^{N,\mathrm{p}})^T]^T. \end{aligned} \tag{3}$$

Let ω^{p} and ν^{p} denote the corresponding process and measurement noise, respectively. Further, we define the following notations

$$\begin{aligned} U &\triangleq [u^{1,\mathrm{p}},\ \ldots,\ u^{N,\mathrm{p}}], & Y &\triangleq [y^{1,\mathrm{p}},\ \ldots,\ y^{N,\mathrm{p}}], \\ \Omega &\triangleq [\omega^{1,\mathrm{p}},\ \ldots,\ \omega^{N,\mathrm{p}}], & V &\triangleq [\nu^{1,\mathrm{p}},\ \ldots,\ \nu^{N,\mathrm{p}}], \\ X &\triangleq [x_{k_1},\ \ldots,\ x_{k_N}]. \end{aligned} \tag{4}$$

It follows from (1) that

$$Y = GX + FU + H\Omega + V, \tag{5}$$

where $F = H(I_L \otimes B)$,

$$G = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^L \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C & 0 & & 0 \\ CA & C & & 0 \\ \vdots & & \ddots & \vdots \\ CA^{L-1} & CA^{L-2} & \cdots & C \end{bmatrix}.$$

Assumption 2. $L \geq \max\{n, m, p\}$.

Remark 1. The conditions $L \geq m$ and $L \geq p$ are assumed only for notational simplicity, while the condition $L \geq n$ is assumed to ensure the observability matrix has full column rank (Chen, 1984, Theorem 6.DO1).

Assumption 3. $\operatorname{rank}\begin{bmatrix} X \\ U \end{bmatrix} = n + Lm.$

Motivating Example: Continuous stirred-tank reactors (CSTRs) play a crucial role in industrial processes, particularly in chemical production and reaction engineering. State monitoring of CSTRs is essential for ensuring process efficiency, maintaining product quality, and preventing safety hazards. According to Bequette (2003), a CSTR for an exothermic reaction can be described by:

$$\begin{aligned} \dot{C}_{\mathrm{A}} &= \frac{q}{V}(C_{\mathrm{f}} - C_{\mathrm{A}}) - k_0 \exp\Big(-\frac{E}{RT}\Big)C_{\mathrm{A}}, \\ \dot{T} &= \frac{q}{V}(T_{\mathrm{f}} - T) - \frac{k_0 \Delta H}{\rho C_{\mathrm{p}}} \exp\Big(-\frac{E}{RT}\Big)C_{\mathrm{A}} + \frac{UA(T_{\mathrm{c}} - T)}{V \rho C_{\mathrm{p}}}, \end{aligned}$$

where $C_{\mathrm{A}} \in \mathbb{R}$ is the reactant concentration; $T \in \mathbb{R}$ is the reactor temperature that can be measured; $T_{\mathrm{c}} \in \mathbb{R}$ is the temperature of the coolant stream that can be manipulated; and see Bequette (2003) for the definitions of other parameters. Note that the nominal operating setpoint of the CSTR corresponds to a steady state C^{s}, T^{s}, and $T_{\mathrm{c}}^{\mathrm{s}}$. By letting $x = [C_{\mathrm{A}} - C^{\mathrm{s}},\ T - T^{\mathrm{s}}]^T$, $u = T_{\mathrm{c}}$, and $y = T$ be the state, input, and output vectors, the dynamics of the CSTR near the nominal operating setpoint can be modeled as (1) (Bequette, 2003), where $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^2$ are unknown system matrices. For this system, we can pre-collect an input-state-output trajectory. Specifically, we can measure temperature using sensors at a minute-level frequency, and measure reactant concentration through chemical analysis methods usually at an hourly frequency (Bequette, 2003). By doing so, we obtain a higher-frequency sampled input–output trajectory, denoted by u^{p} and y^{p}, and a lower-frequency sampled state trajectory, denoted by x^{p}. The objective is to estimate the online reactant concentration using the pre-collected system trajectory.

This paper focuses on state estimation for a predefined state of an unknown system, as shown in the example above, requiring prior data related to this state. This differs from subspace identification or Willems' fundamental lemma-based control (Adachi & Wakasa, 2021; Coulson, Lygeros, & Dörfler, 2019; Oymak & Ozay, 2019; Turan & Ferrari-Trecate, 2021), which relies only on input–output data and does not require a specific state-space model. We will also generalize the results to cases without prior state information, applicable to control tasks such as LQG control.

2.3. Problem statement

The objective of this paper is to estimate an online state trajectory of (1) based on the pre-collected data $\{u^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}\}$. The online trajectory is described by

$$\begin{aligned} u_{[0,k]} &\triangleq [u_0^T,\ u_1^T,\ \ldots,\ u_k^T]^T, \\ x_{[0,k+1]} &\triangleq [x_0^T,\ x_1^T,\ \ldots,\ x_{k+1}^T]^T, \\ y_{[1,k+1]} &\triangleq [y_1^T,\ y_2^T,\ \ldots,\ y_{k+1}^T]^T,\ k \in \mathbb{N}, \end{aligned} \tag{6}$$

where $u_{[0,k]}$ and $y_{[1,k+1]}$ are known, as shown in Fig. 1(b).

Problem: For system (1) with unknown A, B, and C, design a filtering algorithm to estimate the state $x_{[0,k+1]}$ defined in (6), using $\{u^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}\}$ and $\{u_{[0,k]}, y_{[1,k+1]}\}$, defined in (2), (3), and (6), respectively, as

$$\hat{x}_{[0,k+1]} = g_{k+1}(u_{[0,k]}, y_{[1,k+1]}, u^{\mathrm{p}}, y^{\mathrm{p}}, \hat{x}_0^{\mathrm{p}}),\ k \in \mathbb{N}, \tag{7}$$

where $\hat{x}_{[0,k+1]} = [\hat{x}_0^T, \ldots, \hat{x}_{k+1}^T]^T$ denotes the estimate of the online $x_{[0,k+1]}$ that corresponds to (1), and $g_{k+1}(\cdot)$ denotes the filtering algorithm. Moreover, the filtering performance should be quantitatively analyzed and compared with the Kalman filter based on known matrices A, B, and C.

3. RDKF design

In this section, we provide a framework for estimating the state in (6) using the pre-collected data $\{u^{\mathrm{p}}, x^{\mathrm{p}}, y^{\mathrm{p}}\}$. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{y^p}$ be the random variables of $\hat{x}_{[0,k+1]}$, $y_{[1,k+1]}$, and y^{p}, respectively. Then, we define a joint probability density function of $\hat{x}_{[0,k+1]}$, $y_{[1,k+1]}$, and y^{p}:

$$\begin{aligned} &f_{\mathbf{x},\mathbf{y},\mathbf{y^p}}(\hat{x}_{[0,k+1]}, y_{[1,k+1]}, y^{\mathrm{p}}) \\ &\triangleq f_{\mathbf{x},\mathbf{y},\mathbf{y^p}}(\mathbf{x} = \hat{x}_{[0,k+1]}, \mathbf{y} = y_{[1,k+1]}, \mathbf{y^p} = y^{\mathrm{p}}). \end{aligned}$$

Lemma 1. *For system (1), the joint probability density function* $f_{\mathbf{x},\mathbf{y},\mathbf{y^p}}(\hat{x}_{[0,k+1]}, y_{[1,k+1]}, y^{\mathrm{p}})$ *is equivalent to*

$$f_{\mathbf{x},\mathbf{y},\mathbf{y^p}}(\hat{x}_{[0,k+1]}, y_{[1,k+1]}, y^{\mathrm{p}}) = \text{constant} \times f_{\text{off}} \times f_{\text{on}},$$

where

$$f_{\text{off}} = \prod_{i=1}^{N}\prod_{h=0}^{L-1} \exp\Big(-\frac{1}{2}(\hat{\omega}_h^i)^T Q^{-1}\hat{\omega}_h^i\Big) \times \prod_{i=1}^{N}\prod_{h=0}^{L} \exp\Big(-\frac{1}{2}(\hat{v}_h^i)^T R^{-1}\hat{v}_h^i\Big),$$

and

$$f_{\text{on}} = \exp\Big(-\frac{1}{2}(\hat{x}_0-\bar{x}_0)^T P_0^{-1}(\hat{x}_0-\bar{x}_0)\Big) \times \prod_{t=0}^{k} \exp\Big(-\frac{1}{2}\hat{\omega}_t^T Q^{-1}\hat{\omega}_t - \frac{1}{2}\hat{v}_{t+1}^T R^{-1}\hat{v}_{t+1}\Big),$$

$\hat{\omega}_{[0,k]} \triangleq [\hat{\omega}_0^T, \ldots, \hat{\omega}_k^T]^T$ *and* $\hat{v}_{[1,k+1]} \triangleq [\hat{v}_1^T, \ldots, \hat{v}_{k+1}^T]^T$ *are variables to approximate the system process and measurement noise, respectively, satisfying*

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + Bu_t + \hat{\omega}_t, \\ y_t &= C\hat{x}_t + \hat{v}_t,\ t = 0, 1, \ldots, k+1, \end{aligned} \tag{8}$$

and

$$\hat{\omega}^{\mathrm{p}} = [(\hat{\omega}^1)^T, \ldots, (\hat{\omega}^N)^T]^T,\ \hat{\omega}^i = [(\hat{\omega}_0^i)^T, \ldots, (\hat{\omega}_{L-1}^i)^T]^T,$$
$$\hat{v}^{\mathrm{p}} = [(\hat{v}^1)^T, \ldots, (\hat{v}^N)^T]^T,\ \hat{v}^i = [(\hat{v}_0^i)^T, \ldots, (\hat{v}_L^i)^T]^T,$$

are variables with their elements satisfying

$$y_{k_i+h} = CA^h x_{k_i} + \sum_{l=1}^{h} CA^{l-1}(Bu_{h-l}^i + \hat{\omega}_{h-l}^i) + \hat{v}_h^i, \tag{9}$$

for all $h = 1, \ldots, L$, *and* $y_{k_i} = Cx_{k_i} + \hat{v}_0^i$.

The proof of Lemma 1 is given in the online version (Duan, Liu, Xing, & Johansson, 2024). According to Rauch, Tung, and Striebel (1965), when the system matrices A, B, and C are known, the minimum mean-square error (MMSE) state estimate for (1) is the optimal solution to an ML optimization problem. In this paper, we consequently perform state estimation for (1) by instead maximizing the likelihood function derived in Lemma 1, i.e.,

$$\begin{aligned} \min_{\substack{\hat{x}_0,\hat{\omega}^{\mathrm{p}},\hat{v}^{\mathrm{p}},\\ \hat{\omega}_{[0,k]},\hat{v}_{[1,k+1]}}} \quad & -\ln f_{\text{off}} - \ln f_{\text{on}} \\ \text{s.t.} \quad & (8) \text{ and } (9). \end{aligned} \tag{$\mathcal{P}_{k+1}^{I}$}$$

Let $\hat{x}_0^*$, $\hat{\omega}_{[0,k]}^*$, $\hat{v}_{[1,k+1]}^*$, $\hat{\omega}^{\mathrm{p}*}$, and $\hat{v}^{\mathrm{p}*}$ denote the solution to $\mathcal{P}_{k+1}^{I}$. Let $\hat{x}_{[0,k+1]}^* = [(\hat{x}_0^*)^T, (\hat{x}_2^*)^T, \ldots, (\hat{x}_{k+1}^*)^T]^T$ denote the state evolution of (8) given these optimal variables.

Note that the optimization problem $\mathcal{P}_{k+1}^{I}$ is nonlinear and non-convex, and cannot be solved in general. We slightly modify $\mathcal{P}_{k+1}^{I}$ to get a problem solvable with a recursive algorithm and with a solution close to the optimal one with known A, B, and C. Specifically, we modify $\mathcal{P}_{k+1}^{I}$ as

$$\begin{aligned} \min_{\substack{\hat{x}_0,\hat{\omega}_{[0,k]},\\ \hat{v}_{[1,k+1]}}} \quad & \max_{\substack{\hat{\omega}^{\mathrm{p}}\in B_{\epsilon_1}(\hat{\omega}^{\mathrm{p}^*}),\\ \hat{v}^{\mathrm{p}}\in B_{\epsilon_2}(\hat{v}^{\mathrm{p}^*})}} \quad -\ln f_{\text{on}} \\ \text{s.t.} \quad & (8), \end{aligned} \tag{$\mathcal{P}_{k+1}^{II}$}$$

with $B_{\epsilon_1}(\hat{\omega}^{\mathrm{p}^*}) = \{\hat{\omega}^{\mathrm{p}} | \|\hat{\omega}^{\mathrm{p}} - \hat{\omega}^{\mathrm{p}^*}\|_2 \le \epsilon_1\}$ and $B_{\epsilon_2}(\hat{v}^{\mathrm{p}^*}) = \{\hat{v}^{\mathrm{p}} | \|\hat{v}^{\mathrm{p}} - \hat{v}^{\mathrm{p}^*}\|_2 \le \epsilon_2\}$, where $\hat{\omega}^{\mathrm{p}*}$ and $\hat{v}^{\mathrm{p}*}$ are solution to

$$\min_{\hat{\omega}^{\mathrm{p}},\hat{v}^{\mathrm{p}}} \ -\ln f_{\text{off}}, \quad \text{s.t. } (9), \tag{$\mathcal{P}_0^{II}$}$$

Algorithm 1 RDKF

Input: U, Y, X, u, and y, defined in (4) and (6);
Output: $\hat{x}_k$, $k \in \mathbb{N}$;

1: compute matrices

$$A_\sharp = G_{1,\sharp}^\dagger G_{2,\sharp},\ B_\sharp = G_{1,\sharp}^\dagger G_{3,\sharp},\ C_\sharp = G_{4,\sharp}; \tag{10}$$

with

$$\begin{aligned} &G_{1,\sharp} = Z_\sharp(1:Lp;\ 1:n), \\ &G_{2,\sharp} = Z_\sharp(p+1:Lp+p;\ 1:n), \\ &G_{3,\sharp} = Z_\sharp(p+1:Lp+p;\ n+1:2n), \\ &G_{4,\sharp} = G_{1,\sharp}(1:p;\ 1:n),\ Z_\sharp = Y([X^T, U^T]^T)^\dagger; \end{aligned} \tag{11}$$

2: **for** $k = 0, 1, \ldots$ **do**

$$\begin{aligned} &\hat{x}_{k+1|k} = \hat{A}_k\hat{x}_k + \hat{B}_k u_k, \\ &\hat{x}_{k+1} = \hat{x}_{k+1|k} + L_{k+1}(y_{k+1} - \hat{C}_k\hat{x}_{k+1|k}), \\ &L_{k+1} = \bar{P}_{k+1}\hat{C}_k^T(\hat{R} + \hat{C}_k\bar{P}_{k+1}\hat{C}_k^T)^{-1}, \\ &\bar{P}_{k+1} = A_\sharp\hat{P}_k A_\sharp^T + Q, \\ &P_{k+1} = \bar{P}_{k+1} - \bar{P}_{k+1}\hat{C}_k^T(\hat{R} + \hat{C}_k\bar{P}_{k+1}\hat{C}_k^T)^{-1}\hat{C}_k\bar{P}_{k+1}, \end{aligned} \tag{12}$$

where the filter parameters are defined as

$$\hat{A}_k = A_\sharp(I - \lambda\psi_A^2\epsilon\hat{P}_k),\ \hat{B}_k = B_\sharp - \lambda\psi_B^2\epsilon\hat{P}_k,\ \hat{C}_k = C_\sharp,$$
$$\hat{R} = R - \lambda^{-1}\epsilon C_\sharp C_\sharp^T,\ \hat{P}_k = (P_k^{-1} + \lambda\psi_A^2\epsilon I)^{-1},$$

$\hat{x}_{k+1}$ is the estimate of x_{k+1} in (6), $\hat{x}_0 = \bar{x}_0$, λ is any scalar greater than $\|\epsilon C_\sharp^T R^{-1} C_\sharp\|_2$; and ψ_A, ψ_B, and ϵ are given in (19).

3: **end for**

and ϵ_1 and ϵ_2 are the upper bounds of $\|\omega^{\mathrm{p}} - \hat{\omega}^{\mathrm{p}^*}\|_2$ and $\|v^{\mathrm{p}} - \hat{v}^{\mathrm{p}^*}\|_2$, respectively.

To solve $\mathcal{P}_{k+1}^{II}$, we propose the RDKF algorithm. Let us explain the main stages of the algorithm. First, when regarding A, B, and C as unknown variables, the optimal solution to $\mathcal{P}_0^{II}$ can be directly derived as $\hat{\omega}^{\mathrm{p}} = 0$ and $\hat{v}^{\mathrm{p}} = 0$. Then, if Assumption 3 holds, substituting the solution into (9) gives

$$Z_\sharp \triangleq [G_\sharp \quad F_\sharp] = Y([X^T, U^T]^T)^\dagger, \tag{13}$$

where Y, X, and U are given in (5), and $G_\sharp$ and $F_\sharp$ are the estimates of G and F, respectively. Let $A_\sharp$, $B_\sharp$, and $C_\sharp$ be the estimates of A, B, and C, respectively. We assume that $G_\sharp$ and $F_\sharp$ have the same structures with respect to $A_\sharp$, $B_\sharp$ and $C_\sharp$ as G and F have with respect to A, B and C, respectively. Hence, the following equations

$$G_{1,\sharp}A_\sharp = G_{2,\sharp},\ G_{1,\sharp}B_\sharp = G_{3,\sharp},\ C_\sharp = G_{1,\sharp}(1:m;\ 1:n),$$

hold. If $G_{1,\sharp}$ has full column rank, we obtain 10. When there exist positive scalars ψ_A, ψ_B, and ϵ such that $\|A - A_\sharp\|_2 \le \psi_A\epsilon$, $\|B - B_\sharp\|_2 \le \psi_B\epsilon$, and $\|C - C_\sharp\|_2 \le \epsilon$, $\mathcal{P}_{k+1}^{II}$ can be slightly modified as

$$\begin{aligned} \min_{\hat{\omega}_{[0,k]},\hat{v}_{[1,k+1]}} \quad & \max_{\substack{\|\Delta_A\|_2\le\psi_A\epsilon,\\ \|\Delta_B\|_2\le\psi_B\epsilon,\|\Delta_C\|_2\le\epsilon}} \quad -\ln f_{\text{on}} \\ \text{s.t.} \quad & \hat{x}_{t+1} = (A_\sharp + \Delta_A)\hat{x}_t + (B_\sharp + \Delta_B)u_t + \hat{\omega}_t, \\ & y_t = (C_\sharp + \Delta_C)\hat{x}_t + \hat{v}_t,\ t = 0, 1, \ldots, k+1. \end{aligned}$$

By utilizing the regularized least-squares method in Sayed (2001), an explicit solution to the above optimization problem is derived as 12 in Algorithm 1.

Remark 2. Maximum likelihood estimation is a fundamental principle for state estimation, e.g., in switching (Alessandri, Baglietto, & Battistelli, 2010), time-varying (Poncela, Poncela, & Perán,

2013), and nonlinear systems (Marelli, Fu, & Ninness, 2015). This paper employs this principle to deal with state estimation of unknown systems, where the new likelihood function derived in Lemma 1 incorporates both the pre-collected data and the online measurements.

When no prior state can be collected, the RDKF can be directly extended to estimate the state of a balanced realization, applicable to control tasks like LQG control. To move on, we restack U and Y in (4) by replacing L with $2L$, and modify Assumption 3 as

Assumption 4. $\text{rank}(U) = 2Lm$.

Similarly to system identification (Tsiamis et al., 2020; Zheng & Li, 2020), we assume the initial state of pre-collected input–output trajectory (3) to be either zero or unknown zero-mean random noise. In this case, the RDKF proposed in Algorithm 1 remains valid, except that $Z_\sharp$ in 11 is replaced by

$$Z_\sharp^{\text{new}} = [G_\sharp^{\text{new}}\ F_\sharp^{\text{new}}(1:Lp+p;1:Lm)],\ F_\sharp^{\text{new}} = YU^\dagger,$$
$$G_\sharp^{\text{new}} = U_1^s\Sigma_1^{1/2},\ F_{\sharp,H}^{\text{new}} \overset{\text{SVD}}{=\!=} [U_1^s\ U_2^s]\begin{bmatrix}\Sigma_1 & \\ & \Sigma_2\end{bmatrix}\begin{bmatrix}V_1^s \\ V_2^s\end{bmatrix}, \quad (14)$$

where $F_{\sharp,H}^{\text{new}}$ is the Hankel form of $F_\sharp^{\text{new}}$ (Oymak & Ozay, 2019; Tsiamis & Pappas, 2019); "SVD" denotes the singular value decomposition; $\Sigma_1 \in \mathbb{R}^{n\times n}$ contains the n-largest singular values. Let us explain the derivation process of the new result as follows. Similarly to cases with prior states, we still solve $\mathcal{P}_k^{II}$ to design the RDKF for cases without prior states. In the new case, (13) reduces to $F_\sharp^{\text{new}} = YU^\dagger$, where $F_\sharp^{\text{new}}$ is the estimate of F. Note that F is a block-Toeplitz matrix, which can be reshaped as a Hankel matrix F_H satisfying $F_H = GG_B$, where G is the observability matrix and G_B is the controllability matrix (Tsiamis & Pappas, 2019, Section 3). Hence, we can reshape $F_\sharp^{\text{new}}$ to get a Hankel matrix $F_{\sharp,H}^{\text{new}}$. By taking the SVD of $F_{\sharp,H}^{\text{new}}$, the controllability matrix of a balanced realization can be derived as $G_\sharp^{\text{new}}$ in (14). As a result, $Z_\sharp$ in 11 is replaced by $Z_\sharp^{\text{new}}$ in (14), while the other steps in Algorithm 1 remain valid.

4. Data informativity analysis

This section is aimed at analyzing the informativity of the pre-collected data required for performing the RDKF. Similarly to (13), let $Z \triangleq [G\ F]$, and $e_Z = Z_\sharp - Z$ be the error between Z and $Z_\sharp$. It follows from (5) that

$$e_Z = (H\Omega + V)\begin{bmatrix}X \\ U\end{bmatrix}^T\left(\begin{bmatrix}X \\ U\end{bmatrix}\begin{bmatrix}X \\ U\end{bmatrix}^T\right)^{-1}, \quad (15)$$

which is obtained by substituting the explicit expression of $([X^T, U^T]^T)^\dagger$. Similarly to Tsiamis et al. (2020), we assume that the system is stable and $L_i \gg L$. In the experiments of generating data, let $u_h^i \sim \mathcal{N}(0,\ S)$ with $S \in \mathbb{R}^{m\times m} > 0$. In this case, x_{k_i} is a Gaussian variable, and we denote $x_{k_i} \sim \mathcal{N}(0,\ P_x)$ with $P_x \in \mathbb{R}^{n\times n} > 0$. When $L_i \gg L$ and the system state matrix A is stable, it is reasonable to assume that u_h^i and x_{k_i} are mutually uncorrelated, $\forall h = k_i, k_i+1, \ldots, k_i+L-1, \forall i \in \mathcal{V}$. Let $Q = \sigma_\omega^2 I_n$, $R = \sigma_\nu^2 I_p$, $P_\xi = \sigma_\xi^2 I_n$, $S = \sigma_u^2 I_m$, and $P_x = \sigma_x^2 I_n$ for notational simplicity.

Proposition 1. *Consider system (1) with the collected data U, Y, and X defined in (4). Suppose Assumption 3 holds. If $N \geq 32L^2\log(27/\delta)$, then $\|e_Z\|_2 \leq \mathcal{O}(\sqrt{1/N})$ holds with probability at least $1-\delta$ with $\delta \in (0,\ 1)$.*

The proof of Proposition 1 is given in Appendix A.1. Let G_1, G_2, and G_3 represent the corresponding block matrices in Z as $G_{1,\sharp}$, $G_{2,\sharp}$, and $G_{3,\sharp}$ in $Z_\sharp$, respectively, where $G_{1,\sharp}$, $G_{2,\sharp}$, and $G_{3,\sharp}$ are defined in Algorithm 1. Subsequently, we have the following corollary.

Corollary 1. *Consider system (1) with the pre-collected data U, Y, and X, defined in (4). Suppose Assumption 3 holds. For any scalars $\epsilon \in (0,\ 1)$ and $\delta \in (0,\ 1)$, there always exists a positive integer $N_0(\epsilon,\delta)$ such that if $N \geq N_0(\epsilon,\delta)$, then*

$$\|G_1 - G_{1,\sharp}\|_2 \leq \epsilon,\ \|G_2 - G_{2,\sharp}\|_2 \leq \epsilon,\ \|G_3 - G_{3,\sharp}\|_2 \leq \epsilon,$$

hold with probability at least $1-\delta$.

From the proof of Proposition 1, one feasible $N_0(\epsilon,\delta)$ is given by

$$N_0(\epsilon,\delta) = 32L^2\log(27/\delta) \times \max\{1,\ 2M_Z^2/\epsilon^2 + 2\alpha_0^2\}, \quad (16)$$

where M_Z is a positive constant defined as

$$M_Z = \alpha_0\|G_\sharp\|_2 + \beta_0, \quad (17)$$

with $\alpha_0 = \sigma_{\max}\sigma_\omega L/\sigma_{\min}^2$, $\beta_0 = \sigma_{\max}\sigma_\nu/\sigma_{\min}^2$, $\sigma_{\max} = \max\{\sigma_x, \sigma_u\}$, and $\sigma_{\min} = \min\{\sigma_x, \sigma_u\}$. The above result reveals an explicit relation between the estimation error bound ϵ and the number of the pre-collected states N. Based on Corollary 1, we can determine the number N required for ensuring the full column rank of $G_{1,\sharp}$ for performing the proposed RDKF.

Theorem 1. *Suppose Assumptions 1, 2 and 3 hold. If $N \geq N_0(\epsilon,\delta)$ and $\epsilon < \epsilon_0$, where $N_0(\epsilon,\delta)$ is defined in (16) and ϵ_0 is defined as*

$$\epsilon_0 \triangleq \sqrt{\|G_1\|_2^2 + \lambda_{\min}(G_1^TG_1)} - \|G_1\|_2, \quad (18)$$

then $G_{1,\sharp}$ has full column rank with probability at least $1-\delta$.

Theorem 1 is proved in Appendix A.2. Theorem 1 reveals that $G_{1,\sharp}$ has full column rank with any high probability if conditions in Theorem 1 are met.

5. RDKF performance evaluation

This section presents the performance gap between the proposed RDKF and the optimal Kalman filter based on known system matrices.

Theorem 2. *Suppose that Assumptions 1, 2, and 3 hold. When $N \geq N_0(\epsilon,\delta)$ and $\epsilon < \epsilon_0$ with $N_0(\epsilon,\delta)$ and ϵ_0 being defined in (16) and (18), respectively, then*

$$\|A - A_\sharp\|_2 \leq \psi_A\epsilon,\ \|B - B_\sharp\|_2 \leq \psi_B\epsilon,\ \|C - C_\sharp\|_2 \leq \epsilon,$$

simultaneously hold with probability at least $1-\delta$, where ψ_A and ψ_B are positive constants defined in (19).

Theorem 2 is proved in Appendix A.3. Theorem 2 offers the upper bounds for the estimation errors of system matrices. Considering a given $N \geq L_0 \triangleq 32L^2\log(27/\delta)$, it follows from (16) and the proof of Theorem 2 that the values of ϵ, ψ_A, and ψ_B is given by

$$\begin{aligned}\epsilon &= \sqrt{2M_Z^2L_0/(N-2\alpha_0^2L_0)},\\ \psi_A &= \|G_{1,\sharp}^\dagger\|_2(\|A_\sharp\|_2+1)/(1-\|G_{1,\sharp}^\dagger\epsilon\|_2),\\ \psi_B &= \|G_{1,\sharp}^\dagger\|_2(\|B_\sharp\|_2+1)/(1-\|G_{1,\sharp}^\dagger\epsilon\|_2),\end{aligned} \quad (19)$$

which are used for the RDKF presented in Algorithm 1.

According to Theorems 1 and 2, $G_{1,\sharp}$ has full column rank with probability at least $1-\delta$ when $N \geq N_0(\epsilon,\delta)$ and $\epsilon < \epsilon_0$. Hence, $(C,\ A_\sharp)$ is observable (Chen, 1984, Theorem 6.DO1). Further, $\hat{P}_k$, $\bar{P}_k$, and P_k in 12 exponentially converge to the unique solution to Sayed (2001)

$$\begin{aligned}\hat{P}_\sharp &= (P_\sharp^{-1} + \lambda\psi_A^2\epsilon)^{-1},\ \bar{P}_\sharp = A_\sharp\hat{P}_\sharp A_\sharp^T + Q,\\ P_\sharp &= \bar{P}_\sharp - \bar{P}_\sharp C_\sharp^T(\hat{R} + C_\sharp\bar{P}_\sharp C_\sharp^T)^{-1}C_\sharp\bar{P}_\sharp,\end{aligned} \quad (20)$$

with probability at least $1-\delta$ when $N \geq N_0(\epsilon,\delta)$ and $\epsilon < \epsilon_0$. We assume that the filter parameters in 12 are in their steady states for notational simplicity. In addition, the steady-state Kalman filter for (1) with known A, B, and C is given by Anderson and Moore (2005)

$$\begin{aligned}\hat{x}_{k+1|k} &= A\hat{x}_k + Bu_k,\\ \hat{x}_{k+1} &= \hat{x}_{k+1|k} + L(y_{k+1} - C\hat{x}_{k+1|k}),\\ L &= \bar{P}C^T(R + C\bar{P}C^T)^{-1},\\ \bar{P} &= APA^T + Q,\\ P &= \bar{P} - \bar{P}C^T(R + C\bar{P}C^T)^{-1}C\bar{P},\end{aligned} \tag{21}$$

where $\hat{x}_{k+1|k}$ and $\hat{x}_{k+1}$ are the *a priori* and *a posteriori* MMSE estimates, respectively; and L, $\bar{P}$, and P are filter parameters. Further, an assumption on the state x_k is needed, which holds for many systems with closed-loop controllers.

Assumption 5. There exists a positive definite matrix Π such that $\mathbb{E}\{x_k x_k^T\} \leq \Pi$, $\forall k \in \mathbb{N}$.

Let $e_k = \hat{x}_k - x_k$ be the estimation error of the RDKF at step k, and $P_{e,k} \triangleq \mathbb{E}\{e_k e_k^T\}$. A result about the filtering performance of the proposed RDKF is given as follows.

Theorem 3. *Consider system (1) with the pre-collected data U, Y, and X, defined in (4). Suppose Assumptions 1, 2, 3, and 5 hold. When $N \geq N_0(\epsilon,\delta)$ and $\epsilon < \epsilon_0$ with $N_0(\epsilon,\delta)$ and ϵ_0 being defined in (16) and (18), respectively, then*

$$\|P_{e,\infty} - P\|_2 \leq \mathcal{O}\Big(\sqrt{1/N}\Big),$$

holds with probability at least $1-\delta$.

A brief outline of the proof for Theorem 3 is given below. First, it follows from Theorem 2 and (20) that

$$\|\hat{A} - A\|_2,\ \|\hat{B} - B\|_2,\ \|\hat{C} - C\|_2,\ \|\hat{R} - R\|_2,\ \|P_\sharp - \hat{P}_\sharp\|_2,$$

are upper bounded by $\mathcal{O}(\sqrt{1/N})$ with probability at least $1-\delta$. In the following, the proof process is divided into five steps by considering five cases:

Case (1): $\|\hat{B} - B\|_2 = \|\hat{C} - C\|_2 = \|\hat{R} - R\|_2 = \|P_\sharp - \hat{P}_\sharp\|_2 = 0$.
Case (2): $\|\hat{C} - C\|_2 = \|\hat{R} - R\|_2 = \|P_\sharp - \hat{P}_\sharp\|_2 = 0$;
Case (3): $\|\hat{R} - R\|_2 = \|P_\sharp - \hat{P}_\sharp\|_2 = 0$;
Case (4): $\|P_\sharp - \hat{P}_\sharp\|_2 = 0$;
Case (5): The original case in Theorem 3.

To distinguish, let $P_{e,\infty}$ in Cases 1–5 be $P^1_{e,\infty}$–$P^5_{e,\infty}$, respectively. By leveraging the uniform boundedness of filter parameters in each case, we can prove that $\|P^1_{e,\infty} - P\|_2$, $\|P^2_{e,\infty} - P^1_{e,\infty}\|_2$, $\|P^3_{e,\infty} - P^2_{e,\infty}\|_2$, $\|P^4_{e,\infty} - P^3_{e,\infty}\|_2$, and $\|P^5_{e,\infty} - P^4_{e,\infty}\|_2$ are upper bounded by $\mathcal{O}(\sqrt{1/N})$. Hence, we have $\|P^5_{e,\infty} - P\|_2 \leq \mathcal{O}(\sqrt{1/N})$, as stated in Theorem 3. A complete proof of Theorem 3 is available in the online version (Duan et al., 2024).

If no prior state is available, as derived in Section 3, we modify Algorithm 1 by replacing $Z_\sharp$ in 11 with $Z_\sharp^{\text{new}}$ in (14). In this case, the state estimate corresponds to a balanced realization, which can be used for control tasks like LQG control. Similarly to Theorem 2, we can also derive sample-complexity upper bounds for the learning errors of A, B, C. By denoting P in (21) for this case as P_{b}, we have the following corollary.

Corollary 2. *Consider system (1) with the pre-collected data U and Y defined in (4). Suppose Assumptions 1, 2, 4, and 5 hold. In addition, $Z_\sharp$ in Algorithm 1 is replaced with $Z_\sharp^{\text{new}}$ in (14). For any scalars $\epsilon \in (0,\ 1)$ and $\delta \in (0,\ 1)$, there always exists a positive integer $N_0(\epsilon,\delta)$ such that if $N \geq N_0(\epsilon,\delta)$, then*

$$\|P_{e,\infty} - P_{\text{b}}\|_2 \leq \mathcal{O}\Big(\sqrt{1/N}\Big),$$

holds with probability at least $1-\delta$.

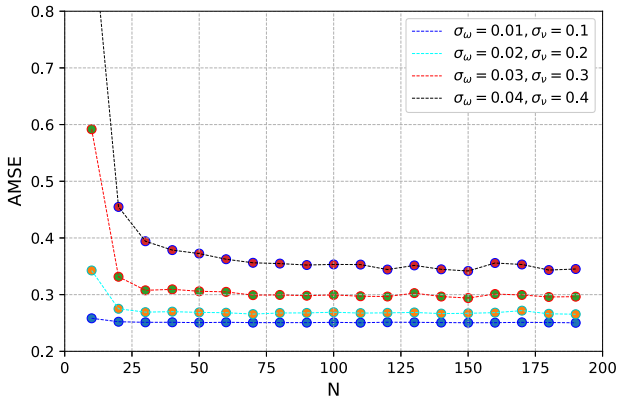


Fig. 2. The values of AMSE under different numbers of samples and magnitudes of noise using the proposed RDKF, where the number N is set as $10, 20, \ldots, 190$, respectively.

Remark 3. Corollary 2 reveals that the designed RDKF can be adapted for cases where prior states are unavailable, while ensuring a sample complexity bound for the filtering performance. In particular, we introduce a novel approach that uses the estimation error covariance as the metric for the sample complexity analysis. This metric offers an intuitive assessment of the performance gap between the proposed RDKF and the optimal Kalman filter based on known system matrices.

6. Simulation

In this section, the effectiveness of the proposed RDKF is illustrated using a simulation of a CSTR (Bequette, 2003). The dynamics of the CSTR is described by a linear state-space model, see Section 2.2. Similarly to Bequette (2003), consider that the nominal operating setpoint of the CSTR corresponds to a steady state $C^s = 0.5$ mol/l, $T^s = 350$ K, and $T_c^s = 300$ K. Let $x = [C_A - C^s,\ T - T^s]^T$, $u = T_c$, and $y = T$ be the state, input, and output vectors. By utilizing a sampling time of $t_s = 0.1$ min, the dynamics of the CSTR is modeled as (1) (Sui, Johansen, & Feng, 2010) with

$$A = \begin{bmatrix} 0.7776 & -0.0045 \\ 26.6186 & 1.8555 \end{bmatrix},\ B = \begin{bmatrix} -0.0004 \\ 0.2907 \end{bmatrix},\ C = [0 \quad 1].$$

Let $\sigma_\omega = 0.01$ and $\sigma_\nu = 0.1$. Assume that the above matrices A and B are unknown. Suppose that we can collect an input-state-output sequence u^{p}, x^{p}, and y^{p} defined in (2) and (3), where the parameters are chosen as $N = 100$, $L = 5$, $\sigma_x = 0.4$, and $\sigma_u = 2$. Based on these data, we apply the proposed RDKF to estimate an online trajectory of the CSTR with $x_0 = [0.4,\ 5]^T$ and $u_k = -8y_k$. To proceed, two types of errors are defined:

$$\text{MSE}(h) = \frac{1}{50}\sum_{k=1}^{50}\|x_k^h - \hat{x}_k^h\|_2^2,\ \text{AMSE} = \frac{1}{N_t}\sum_{h=1}^{N_t}\text{MSE}(h),$$

where the notation h denotes the hth trial, and $N_t = 200$ denotes the number of Monte Carlo trials.

The filtering performance of the CSTR by the designed RDKF is illustrated in Figs. 2 and 3. Fig. 2 illustrates that the filtering performance of the proposed RDKF is improved with an increasing number of samples, which coincides with Theorem 3. Moreover, it also shows that the smaller noise magnitudes leads to a better filtering performance. In addition, four relevant filtering methods in the literature are included for comparison, namely the model-based Kalman filter using known system matrices (MBKF) (Anderson & Moore, 2005), system identification-based

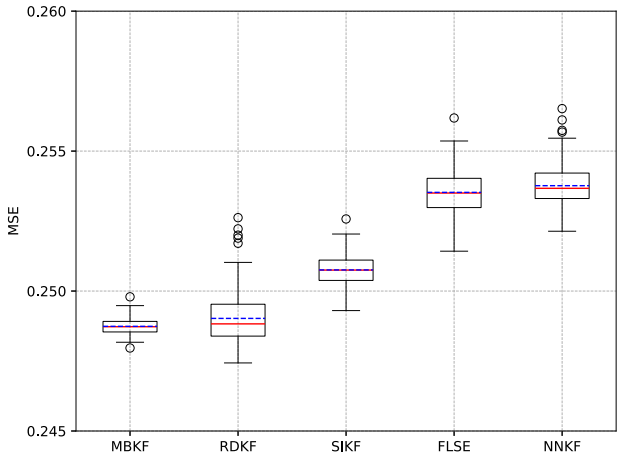


Fig. 3. The values of MSE of 200 Monte Carlo trials using different filtering methods, where the red solid line and the blue dotted line denote the median and mean, respectively; the tops and bottoms of each box represent the 25th and 75th percentiles, respectively; and black circles denote outliers beyond 1.5 times the interquartile range.

Kalman filter (SIKF) (Verhaegen & Verdult, 2007), Willems' fundamental lemma-based state estimator (FLSE) (Liu et al., 2024), and NNs-based Kalman filter (NNKF) (Revach et al., 2022). It is worth mentioning that the SIKF, FLSE, and NNKF use a prior state trajectory sampled at every time step, whereas the RDKF employs a lower-frequency sampled state trajectory. Even so, it is shown in Fig. 3 that the RDKF demonstrates better filtering performance and superior capability in handling noisy data. The above observations illustrate the theoretical results obtained in this paper.

7. Conclusion

This paper proposed a new RDKF for a class of unknown linear systems, where a prior input–output trajectory sampled at a higher frequency and a prior state trajectory sampled at a lower frequency are available. Specifically, an ML optimization problem has been formulated and solved to construct the state estimate. A sample-complexity upper bound has been derived for the performance gap between the designed RDKF and the Kalman filter with known system parameters. Simulations have demonstrated the effectiveness of the theoretical results. In the future, we will further explore data-driven unknown input observers, leveraging the solid theoretical foundation laid by Lv, Li, and Duan (2024), which presented an innovative and practically applicable framework for minimal-order fixed-time unknown input observers.

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Appendix

A.1. Proof of Proposition 1

First, according to (15), it can be directly derived that

$$\|e_Z\|_2 \le (\|H\Omega[X^T, U^T]\|_2 + \|V[X^T, U^T]\|_2) \times \|([X^T, U^T]^T[X^T, U^T])^{-1}\|_2. \tag{22}$$

In the following, the three terms on the right side of (22) are analyzed. For the first term, with probability at least $1-\delta/3$,

$$\|H\Omega[X^T, U^T]\|_2 \le \|H\|_2\|\Omega[X^T, U^T]\|_2 \le 4\|H\|_2\sigma_\omega\sigma_{\max}\sqrt{N(n+Ln+Lm)\log(27/\delta)},$$

holds when $N \ge 2(n+Ln+Lm)\log(4/\delta)$, where $\sigma_{\max} = \max\{\sigma_x, \sigma_u\}$ and the second "$\le$" is derived using Lemma 1 in Dean, Mania, Matni, Recht, and Tu (2020), Zheng and Li (2020). Similarly, the second term

$$\left\|V[X^T, U^T]\right\|_2 \le 4\sigma_v\sigma_{\max}\sqrt{N(n+p+Lp+Lm)\log(27/\delta)},$$

holds with probability at least $1-\delta/3$ when $N \ge 2(n+p+Lp+Lm)\log(3/\delta)$. By applying Lemma 2 in Dean et al. (2020), Zheng and Li (2020), the last term

$$\sqrt{\lambda_{\min}([X^T, U^T]^T[X^T, U^T])} \ge \sigma_{\min}\left(\sqrt{N} - \sqrt{n+Lm} - \sqrt{2\log(3/\delta)}\right) \ge \frac{1}{2}\sigma_{\min}\sqrt{N},$$

holds with probability at least $1-\delta/3$, where $\sigma_{\min} = \min\{\sigma_x, \sigma_u\}$, and the second "$\ge$" holds when $N \ge 8(n+Lm)+16\log(3/\delta)$. Hence, the last term on the right side of (22) can be relaxed as

$$\|([X^T, U^T]^T[X^T, U^T])^{-1}\|_2 \le 4/(N\sigma_{\min}^2).$$

Using the union bound, it can be derived that

$$\|e_Z\|_2 < M_Z\sqrt{\log(27/\delta)/N},$$

holds with probability at least $1-\delta$ when $N \ge 8(n+Lm)+2(Lp+Lm+Ln+p+n+3)\log(3/\delta)$, where M_Z is a constant defined below (16), and Assumption 2 is used such that $2n+Lm \le L(2+m)$, $n+Ln+Lm < L(1+n+m)$ and $n+p+Lp+Lm < L(1+2p+m)$. Noting that we have $\log(10/\delta) \ge 1$, the condition $N \ge 8(n+Lm)+2(Lp+Lm+Ln+p+n+3)\log(3/\delta)$ can be further relaxed as $N \ge 32L^2\log(10/\delta)$. Now, the proof of Proposition 1 is complete.

A.2. Proof of Theorem 1

First of all, note that

$$\begin{aligned} &G_1^TG_1 - G_{1,\sharp}^TG_{1,\sharp} \\ &= (G_1 - G_{1,\sharp})^TG_1 + [(G_{1,\sharp} - G_1) + G_1]^T(G_1 - G_{1,\sharp}) \\ &\le \|G_1 - G_{1,\sharp}\|_2\|G_1\|_2I_n + (\|G_{1,\sharp} - G_1\|_2 + \|G_1\|_2) \times \|G_{1,\sharp} - G_1\|_2I_n. \end{aligned}$$

Then, it follows from Corollary 1 that, with probability at least $1-\delta$, $G_1^TG_1 - G_{1,\sharp}^TG_{1,\sharp} \le \epsilon^2I_n + 2\epsilon\|G_1\|_2I_n$ holds, when $N \ge N_0(\epsilon, \delta)$. Further, we have

$$\begin{aligned} \lambda_{\min}(G_1^TG_1)I_n &\le G_1^TG_1 = G_1^TG_1 - G_{1,\sharp}^TG_{1,\sharp} + G_{1,\sharp}^TG_{1,\sharp} \\ &\le \epsilon^2I_n + 2\epsilon\|G_1\|_2I_n + G_{1,\sharp}^TG_{1,\sharp}. \end{aligned}$$

Besides, according to Chen (1984, Theorem 6.DO1), $G_1^TG_1 > 0$ when Assumptions 1 and 2 hold, which indicates $\lambda_{\min}(G_1^TG_1) > 0$ holds. All together, with probability at least $1-\delta$, $G_{1,\sharp}^TG_{1,\sharp} \ge \left[\lambda_{\min}(G_1^TG_1) - \epsilon^2 - 2\epsilon\|G_1\|_2\right]I_n > 0$ holds, when $N \ge N_0(\epsilon, \delta)$ and $\epsilon < \sqrt{\|G_1\|_2^2 + \lambda_{\min}(G_1^TG_1)} - \|G_1\|_2$. This ensures that $G_{1,\sharp}$ has full column rank.

A.3. Proof of Theorem 2

It follows from (5) that

$$\begin{aligned} A - A_\sharp &= (G_1^\dagger - G_{1,\sharp}^\dagger)G_1 A + G_{1,\sharp}^\dagger (G_2 - G_{2,\sharp}) \\ &= G_{1,\sharp}^\dagger (G_{1,\sharp} - G_1)A + G_{1,\sharp}^\dagger (G_2 - G_{2,\sharp}), \end{aligned}$$

where the last "=" is based on $G_{1,\sharp}^\dagger G_{1,\sharp} = G_1^\dagger G_1 = I_n$. According to the results revealed in Corollary 1, $\|A - A_\sharp\|_2 \le \|G_{1,\sharp}^\dagger\|_2(\|A\|_2 + 1)\epsilon$ holds when $N \ge N_0(\epsilon, \delta)$ with $\epsilon < \epsilon_0$ in (16). Since $G_{1,\sharp}^\dagger = (G_{1,\sharp}^T G_{1,\sharp})^{-1} G_{1,\sharp}^T$, we have

$$\begin{aligned} \|G_{1,\sharp}^\dagger\|_2 &= \sqrt{\lambda_{\max}\big((G_{1,\sharp}^\dagger)^T G_{1,\sharp}^\dagger\big)} = \sqrt{\lambda_{\max}\big(G_{1,\sharp}^\dagger (G_{1,\sharp}^\dagger)^T\big)} \\ &= \sqrt{\lambda_{\max}\big((G_{1,\sharp}^T G_{1,\sharp})^{-1}\big)} = \sqrt{1/\lambda_{\min}\big(G_{1,\sharp}^T G_{1,\sharp}\big)} \\ &\le \sqrt{1/(\lambda_{\min}(G_1^T G_1) - \epsilon^2 - 2\epsilon\|G_1\|_2)}. \end{aligned}$$

Hence, $\|A\|_2\|G_{1,\sharp}^\dagger\|_2 + \|G_{1,\sharp}^\dagger\|_2$ is uniformly bounded. Similarly, with probability at least $1 - \delta$,

$$\|B - B_\sharp\|_2 \le \|G_{1,\sharp}^\dagger\|_2(\|B\|_2 + 1)\epsilon,$$

and $\|C - G_{4,\sharp}\|_2 \le \|G_1 - G_{1,\sharp}\|_2 \le \epsilon$ hold. Hence, Theorem 2 is proved.

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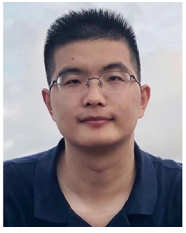

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