Automatica 164 (2024) 111627

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Transient behavior of gossip opinion dynamics with community structure[☆]

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ARTICLE INFO

Article history:
Received 31 May 2022
Received in revised form 27 November 2023
Accepted 14 February 2024
Available online xxxx

Keywords:
Opinion dynamics
Transient behavior
Community structure
Phase transition
Gossip model

ABSTRACT

We study transient behavior of gossip opinion dynamics, in which agents randomly interact pairwise over a weighted graph with two communities. Edges within a community have identical weights different from edge weights between communities. We first derive an upper bound for the second moment of agent opinions. Using this result, we obtain upper bounds for probability that a large proportion of agents have opinions close to average opinions. The results imply a phase transition of transient behavior of the process: When edge weights within communities are larger than those between communities and those between regular and stubborn agents, most agents in the same community hold opinions close to the average opinion of that community with large probability, at an early stage of the process. However, if the difference between intra- and inter-community weights is small, most of the agents instead hold opinions close to everyone's average opinion at the early stage. In contrast, when the influence of stubborn agents is large, agent opinions settle quickly to steady state. We then conduct numerical experiments to validate the theoretical results. Different from traditional asymptotic analysis in most opinion dynamics literature, the paper characterizes the influence of stubborn agents and community structure on the initial phase of the opinion evolution.

1. Introduction

Opinion dynamics studies how personal opinions change through interactions in social networks. Analysis of convergence and stability of opinion dynamics has gained considerable attention in recent decades (Proskurnikov & Tempo, 2018), but less research has focused on transient behavior of the process. Social networks often have topology where subgroups of nodes are densely connected internally but loosely connected with others (that is, community structure, Fortunato & Hric, 2016). Such structure can influence opinion dynamics (Conover et al., 2011; Cota et al., 2019). It is often difficult to determine whether a real social network reaches steady state or not, and whether its communities evolve homogeneously at the early stage. So it is necessary to investigate how the opinion dynamics, especially during its initial phase, corresponds to its community structure. Such results can also provide insight into community detection based on state observations (Schaub et al., 2020; Xing et al., 2023) and model reduction for large-scale networks (Cheng et al., 2018).

1.1. Related work

Individual opinions can be modeled by either continuous or discrete variables (Castellano et al., 2009; Proskurnikov & Tempo, 2018). There are at least three types of continuous-opinion models explaining how interpersonal interactions shape social opinion profiles, namely, models of assimilative, homophily, and negative influences (Flache et al., 2017; Proskurnikov & Tempo, 2018). Evidences for all three types have been found in recent empirical studies (De et al., 2019; Friedkin et al., 2021; Kozitsin, 2023). A crucial example of the first class of models is the DeGroot model (DeGroot, 1974), in which agents update according to the average of their neighbors' opinions. The Friedkin–Johnsen (FJ) model (Friedkin & Johnsen, 1990) generalizes the DeGroot model and allows long-term disagreement, rather than consensus, by assuming that agents are consistently affected by their initial opinions. The Hegselmann–Krause (HK) model (Hegselmann & Krause, 2002) and the Deffuant–Weisbuch (DW) model (Deffuant et al., 2000) are representatives of the second model class, where agents stay away from those holding different beliefs, and tend to form clusters. Negative influences in the third class of models

☆ This work was supported by the Knut and Alice Wallenberg Foundation (Wallenberg Scholar Grant), the Swedish Research Council (Distinguished Professor Grant 2017-01078), and the Swedish Foundation for Strategic Research (CLAS Grant RIT17-0046). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Julien M. Hendrickx under the direction of Editor Christos G. Cassandras.

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https://doi.org/10.1016/j.automatica.2024.111627
0005-1098/

can increase opinion difference, and make the group end in polarization (Proskurnikov & Tempo, 2018; Shi et al., 2019).

Most studies of opinion dynamics have focused on asymptotic behavior, attempting to answer why opinion disagreement occurs in the long run even though social interactions tend to reduce opinion difference (Abelson, 1964; Flache et al., 2017). In contrast, transient behavior has attracted less attention. As the availability of large-scale datasets increases, there is a growing need to understand how the process behaves over a finite time interval (Banisch et al., 2012; Chowell et al., 2016; Noorazar et al., 2020). Important behavioral dynamics, such as election and online discussion, often have finite duration, and their prediction based on transient evolution is of great interest (Banisch & Araújo, 2010; De et al., 2019). Extensive amount of information produced by social media nowadays may change public opinions only temporarily (Hill et al., 2013), making it hard for the dynamical process to reach steady state. Asymptotic analysis thus may not be sufficient for understanding such scenarios. Finally, large-scale networked dynamical processes may converge slowly (Banisch et al., 2012; Lorenz, 2006), but stay close to a certain state for a long time (Barbillon et al., 2015; Dietrich et al., 2016). To distinguish between the two types of states can be challenging and requires knowledge of transient system behavior. Banisch et al. (2012) propose a framework to analyze the transient stage of discrete-opinion models. Barbillon et al. (2015) study quasi-stationary distributions of a contact process, and Xiong et al. (2017) analyze the transient opinion profiles of a voter model. Dietrich et al. (2016) provide criteria for detecting transient clusters in a generalized HK model that normally reaches a consensus asymptotically. Shree et al. (2022) study how opinion difference evolves over finite time intervals for a bounded confidence model.

The study of community structure can be traced back to Festinger (1949), in which a community is defined as a complete subgraph in a network. One of the modern definitions for communities is modularity, which characterizes the nonrandomness of a group partition (Newman & Girvan, 2004). The stochastic block model (SBM) (Abbe, 2017), generating random graphs with communities, is another popular framework in the literature. Researchers have studied how community structure of a network influences opinion evolution, based on several models such as the DW model (Fennell et al., 2021; Gargiulo & Huet, 2010), the Taylor model (Baumann et al., 2020), and the Sznajd model (Si et al., 2009). Como and Fagnani (2016) study the DeGroot model with stubborn agents over a weighted graph, and show that the steady state of the same community concentrates around the state of some stubborn agent.

This paper considers transient behavior of a gossip model with stubborn agents, where agents randomly interact in pairs. The model is a stochastic counterpart of the DeGroot model. It captures the random nature of interpersonal influence and exhibits various behavior. Consensus of the model has been studied by Boyd et al. (2006) and Fagnani and Zampieri (2008). Acemoğlu et al. (2013) show that the existence of stubborn agents may explain opinion fluctuations, and also that regular agents reach similar expected steady state, if the network is highly fluid. Como and Fagnani (2016) show that polarization can emerge for the model over a weighted graph with two stubborn agents. Studying transient behavior of the model can provide insight into analysis of more complex models, because the linear averaging rule is a key building block of most opinion models.

1.2. Contribution

This paper studies transient behavior of the gossip model over a weighted graph with two communities. It is assumed that edges within communities have identical weights different from edge weights between communities. We first obtain an upper bound for the second moment of agent states (Lemma 4). Using this result, we provide probability bounds for agent states concentrating around average opinions (Corollary 1), expected average opinions (Theorem 1), and average initial opinions (Theorem 2). The results reveal a phase transition phenomenon (Holme & Newman, 2006; Shi et al., 2016): When edge weights within communities are larger than those between communities and those between regular and stubborn agents, most agents in the same community have states with small deviation from the average opinion of that community with large probability, at the early stage of the process. If the difference between intra- and inter-community weights is not so large, most agents have states concentrating around everyone's average opinion with small error and large probability (Corollary 2). In contrast, if weights between regular and stubborn agents are larger than those between regular agents, agent states have a distribution close to their stationary one, right after the beginning of the process (Theorem 3 and Corollary 3).

These results indicate that the gossip model has entirely different transient behavior, under different link strength parameters. It is known that the expected steady state depends on the positions of stubborn agents (Acemoğlu et al., 2013), and the model reaches a consensus if there are no stubborn agents (Boyd et al., 2006; Fagnani & Zampieri, 2008). The obtained results show that agents may form transient clusters if the influence of stubborn agents is relatively small. These transient clusters may not be the same as the stationary ones, because they only depend on initial states of regular agents and edge weights. The results also demonstrate how transient behavior corresponds to community structure, by showing that the difference between intra- and inter-community weights has to be large enough to ensure the existence and duration of the corresponding transient clusters.

The obtained results can be directly applied to community detection based on state observations (Schaub et al., 2020; Xing et al., 2023). Suppose that a network is unknown but several snapshots of an opinion dynamic are available. Our analysis ensures that partitioning agent states can recover the community labels of agents, if the states are collected in a transient time interval and intra-community influence is large. The results can also provide insight into predicting and distinguishing transient behavior of an opinion formation process (Banisch & Araújo, 2010; Banisch et al., 2012). For example, when the size of a network and the difference between intra- and inter-community interaction strength are large, the duration of transient behavior is expected to be large as well. Given an estimate of the runtime of a process, we may determine whether the current clusters are steady or not. Finally, exploiting properties of transient clusters can help improve model reduction at the initial phase of dynamics over large-scale networks with community structure (Cheng et al., 2018). If agents in the same community have states close to each other at the early stage, we may track the opinion formation process with much less parameters than the original system, by replacing topological data with community labels.

In the early work (Xing & Johansson, 2023), we study how the expectation of agent states concentrates around average initial states, which follows from Lemma 3 of the current paper. Here we directly investigate how agent states evolve by analyzing the second moment and characterizing transient behavior in detail.

1.3. Outline

Section 2 introduces the model and formulates the problem. Section 3 provides theoretical results. Numerical experiments are presented in Section 4. Section 5 concludes the paper. Some proofs are given in appendices.

Notation. Denote the n-dimensional Euclidean space by $\mathbb{R}^n$, the set of $n \times m$ real matrices by $\mathbb{R}^{n\times m}$, the set of nonnegative integers by $\mathbb{N}$, and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Denote the natural logarithm by $\log x$, $x > 0$. Let $\mathbf{1}_n$ be the all-one vector with dimension n, $e_i^{(n)}$ be the n-dimensional unit vector with ith entry being one, I_n be the $n \times n$ identity matrix, and $\mathbf{0}_{m,n}$ be the $m \times n$ all-zero matrix. Denote the Euclidean norm of a vector by $\|\cdot\|$. For a vector $x \in \mathbb{R}^n$, denote its ith entry by x_i, and for a matrix $A \in \mathbb{R}^{n\times n}$, denote its (i,j)th entry by a_{ij} or $[A]_{ij}$. The cardinality of a set $\mathcal{S}$ is written as $|\mathcal{S}|$. For two sequences of real numbers, $f(n)$ and $g(n) > 0$, $n \in \mathbb{N}$, we write $f(n) = O(g(n))$ if $|f(n)| \le Cg(n)$ for all $n \in \mathbb{N}$ and some $C > 0$, and $f(n) = o(g(n))$ if $|f(n)|/g(n) \to 0$. Further assuming $f(n) > 0$, we denote $f(n) = \omega(g(n))$ if $g(n) = o(f(n))$, $f(n) = \Omega(g(n))$ if there is $C > 0$ such that $f(n) \ge Cg(n)$ for all $n \in \mathbb{N}$, and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold. For $x, y \in \mathbb{R}$, denote $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

2. Problem formulation

The gossip model with stubborn agents is a random process evolving over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V}$ is the node set with $|\mathcal{V}| = n \ge 2$, $\mathcal{E}$ is the edge set, and $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix. The graph $\mathcal{G}$ has no self-loops (i.e., $a_{ii} = 0$, $\forall i \in \mathcal{V}$). $\mathcal{V}$ contains two types of agents, regular and stubborn, denoted by $\mathcal{V}_r$ and $\mathcal{V}_s$, respectively (so $\mathcal{V} = \mathcal{V}_r \cup \mathcal{V}_s$ and $\mathcal{V}_r \cap \mathcal{V}_s = \emptyset$). In this paper we assume that the regular agents form two disjoint communities $\mathcal{V}_{r1}$ and $\mathcal{V}_{r2}$, and denote $\mathcal{C}_i = k$ if $i \in \mathcal{V}_{rk}$, $k = 1, 2$. We call $\mathcal{C}$ the community structure of the graph. Regular agent i has state $X_i(t) \in \mathbb{R}$ at time $t \in \mathbb{N}$, and stubborn agent j has state z_j^s. Stacking these states, we denote the state vector of regular agents at time t by $X(t) \in \mathbb{R}^{n_r}$ and the state vector of stubborn agents by $z^s \in \mathbb{R}^{n_s}$, where $n_r := |\mathcal{V}_r|$ and $n_s := |\mathcal{V}_s|$. The random interaction of the gossip model is captured by an interaction probability matrix $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ satisfying that $w_{ij} = w_{ji} = a_{ij}/\alpha$, where $\alpha = \sum_{i=1}^n \sum_{j=i+1}^n a_{ij}$ is the sum of all edge weights and $\mathbf{1}^T W \mathbf{1}/2 = 1$. At time t, edge $\{i, j\}$ is selected with probability w_{ij} independently of previous updates, and agents update as follows,

$$X(t+1) = Q(t)X(t) + R(t)z^s, \tag{1}$$

where $Q(t) \in \mathbb{R}^{n_r \times n_r}$, $R(t) \in \mathbb{R}^{n_r \times n_s}$, and

$$[Q(t),\ R(t)] = \begin{cases} [I_{n_r} - \frac{1}{2}(e_i^{(n_r)} - e_j^{(n_r)})(e_i^{(n_r)} - e_j^{(n_r)})^T, \quad \mathbf{0}_{n_r,n_s}], & \text{if } i, j \in \mathcal{V}_r, \\ [I_{n_r} - \frac{1}{2}e_i^{(n_r)}(e_i^{(n_r)})^T, \quad \frac{1}{2}e_i^{(n_r)}(e_j^{(n_s)})^T], & \text{if } i \in \mathcal{V}_r, j \in \mathcal{V}_s. \end{cases}$$

That is, only regular agents in $\{i, j\}$ update their states to the average of the two agents' previous states.

We study how community structure and stubborn agents influence transient behavior of agent states $X(t)$. By transient behavior we mean a property of $X(t)$ that holds over a finite time interval, as opposed to asymptotic behavior that holds as time $t \to \infty$. As mentioned in Section 1, agents in the same community tend to have similar states, but how well and how long these clusters form still require rigorous analysis. We characterize the transient clusters of $X(t)$ based on three types of references: (1) agents' average states within communities and everyone's average state at time t, (2) expected average states at time t, and (3) average initial states. As a comparison, we also study the time when the distribution of $X(t)$ is close to the stationary distribution. To sum up, the problem is as follows.

Problem. Given the initial states $X(0)$, the stubborn states z^s, the community structure $\mathcal{C}$, and the weighted adjacency matrix A, provide bounds for the deviation of $X(t)$ from the three types of average states over finite time intervals, and bounds for the time when the distribution of $X(t)$ is close to the stationary distribution.

Probability bounds for the deviation of $X(t)$ from average states at time t are given in Corollary 1, which is a consequence of second-moment analysis (Lemma 4). The deviation from expected average states is analyzed in Theorem 1. Theorem 2 shows concentration of $X(t)$ around average initial states. Theorem 3 gives a lower bound of the time when $X(t)$ is close to steady state.

3. Theoretical analysis

We study transient behavior of gossip model with two communities. The analysis provides crucial insight into understanding transient behavior of the model under general conditions. Main results are presented in Section 3.1 and a discussion is given in Section 3.2.

3.1. Main results

We assume that the regular agents form two disjoint communities $\mathcal{V}_{r1}$ and $\mathcal{V}_{r2}$ with equal size. For simplicity, sort the agents as follows: $\mathcal{V}_{r1} = \{1, \ldots, r_0 n/2\}$, $\mathcal{V}_{r2} = \{1 + r_0 n/2, \ldots, r_0 n\}$, and $\mathcal{V}_s = \{1 + r_0 n, \ldots, n\}$, with $r_0 \in (0, 1)$ such that $r_0 n = n_r$ is an even integer. The proportion of stubborn agents is denoted by $s_0 := 1 - r_0$. We introduce the following assumptions, illustrated in Fig. 1, for the weighted adjacency matrix A of graph $\mathcal{G}$.

Assumption 1 (*Network Topology*).
(i) There exist $l_s^{(r)}, l_d^{(r)} \in (0, 1)$, depending on n, such that $a_{ij} = l_s^{(r)} = l_s^{(r)}(n)$ for $i, j \in \mathcal{V}_r$ with $i \ne j$ and $\mathcal{C}_i = \mathcal{C}_j$, $a_{ij} = l_d^{(r)} = l_d^{(r)}(n)$ for $i, j \in \mathcal{V}_r$ with $\mathcal{C}_i \ne \mathcal{C}_j$.
(ii) There exist $l_{ij}^{(s)} \in [0, 1)$ with $1 \le i \le r_0 n$ and $1 \le j \le s_0 n$, depending on n, such that $a_{i,r_0n+j} = a_{r_0n+j,i} = l_{ij}^{(s)} = l_{ij}^{(s)}(n)$. For $r_0 n + 1 \le i, j \le n$, $a_{ij} = 0$.
(iii) There exists $l^{(s)} \ge 0$ depending on n such that $\sum_{1\le j\le s_0 n} l_{ij}^{(s)} = l^{(s)} = l^{(s)}(n)$ for all $i \in \mathcal{V}_r$.

Remark 1. Assumption 1(i) indicates that the graph on regular agents is a weighted graph, where edges between agents in the same community have the same weight $l_s^{(r)}$ and those between communities have weight $l_d^{(r)}$. In other words, the influence strength between agents depends on their community labels. Such a weighted graph can be treated as the expected adjacency matrix of an SBM, in which nodes are assigned with community labels and are connected by edges with probability depending on their labels. We introduce this simplified assumption to highlight the phase transition phenomenon in the transient phase of the dynamics, and analysis under this assumption is still nontrivial. It is possible to generalize the results to the SBM case by considering the concentration of adjacency matrices (Chung & Radcliffe, 2011). Although assuming that interpersonal influence depends only on community labels is a simplified setting, this model has been found to be effective also in empirical studies (De et al., 2019). Note that the adjacency matrix A has zero entries on the diagonal from the assumption that the graph has no self-loops. Parameter $l^{(s)}$ given in Assumption 1(iii) is the sum of edge weights between one regular agent and all stubborn agents, and thus represents the total influence of stubborn agents on this regular agent. We assume that this sum is the same for all regular agents for analysis simplicity. The results can be extended to the case where the weight sums have upper and lower bounds.

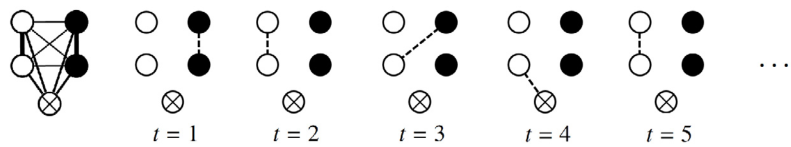


Fig. 1. Illustration of Assumption 1. The graph on the left demonstrates the underlying network with two communities (dots and circles) and one stubborn agent (the circle with a cross). Solid lines represent weighted edges. The weights are indicated by line thickness. Edge weights within communities are larger than between communities ($l_s^{(r)} > l_d^{(r)}$). The edge weights between regular agents and the stubborn agent are the same. The rest of the graphs show random interactions between agents, represented by dashed lines. Agents interact more often if they have an edge with a larger weight.

We further impose an assumption for the initial vector.

Assumption 2 (*Initial Condition*)**.** The initial regular states $X(0)$ and the stubborn states z^s are deterministic, and satisfy that $|X_i(0)| \le c_x$ and $|z_j^s| \le c_x$, for all $1 \le i \le r_0 n$ and $1 \le j \le s_0 n$, and some $c_x > 0$.

From the definitions of $Q(t)$ and $R(t)$, it follows that

$$\bar{Q} := \mathbb{E}\{Q(t)\} = I - \frac{1}{2\alpha}\begin{bmatrix} d_1 & -a_{12} & \cdots & -a_{1,r_0 n} \\ -a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -a_{r_0 n,1} & \cdots & -a_{r_0 n, r_0 n-1} & d_{r_0 n}\end{bmatrix}, \tag{2}$$

$$\bar{R} := \mathbb{E}\{R(t)\} = \frac{1}{2\alpha}\tilde{M} := \frac{1}{2\alpha}\begin{bmatrix} a_{1,r_0 n+1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{r_0 n, r_0 n+1} & \cdots & a_{r_0 n,n}\end{bmatrix}, \tag{3}$$

where $d_i = \sum_{j\in\mathcal{V}} a_{ij},\ i \in \mathcal{V}_r$. Note that Assumption 1 implies $d_i = r_0 n(l_s^{(r)} + l_d^{(r)})/2 + l^{(s)} - l_s^{(r)} =: \bar{d},\ i \in \mathcal{V}_r$, so

$$\bar{Q} = I - \frac{1}{2\alpha}[(\bar{d} + l_s^{(r)})I - \tilde{A}],$$

$$\tilde{A} := \begin{bmatrix} \mathbf{1}_{r_0 n/2} & \mathbf{0}_{r_0 n/2} \\ \mathbf{0}_{r_0 n/2} & \mathbf{1}_{r_0 n/2}\end{bmatrix}\begin{bmatrix} l_s^{(r)} & l_d^{(r)} \\ l_d^{(r)} & l_s^{(r)}\end{bmatrix}\begin{bmatrix} \mathbf{1}_{r_0 n/2} & \mathbf{0}_{r_0 n/2} \\ \mathbf{0}_{r_0 n/2} & \mathbf{1}_{r_0 n/2}\end{bmatrix}^T.$$

It can be shown that $\tilde{A}$ has a simple eigenvalue $r_0 n(l_s^{(r)} + l_d^{(r)})/2$, and a simple eigenvalue $r_0 n(l_s^{(r)} - l_d^{(r)})/2$, and the corresponding unit vectors are $\eta := \mathbf{1}_{r_0 n}/\sqrt{r_0 n}$ and $\xi := [\mathbf{1}_{r_0 n/2}^T \ -\mathbf{1}_{r_0 n/2}^T]/\sqrt{r_0 n}$, respectively. Since $\tilde{A}$ is symmetric, it has the eigenvalue zero with multiplicity $r_0 n - 2$ with orthogonal unit eigenvalues $w^{(i)}$, $3 \le i \le r_0 n$. Moreover, $\eta, \xi, w^{(3)}, \ldots, w^{(r_0 n)}$ form an orthonormal basis of $\mathbb{R}^{r_0 n}$. Denoting $\lambda_1 := l^{(s)}/(2\alpha)$, $\lambda_2 := (l_d^{(r)} r_0 n + l^{(s)})/(2\alpha)$, and $\lambda_3 := [(l_s^{(r)} + l_d^{(r)})r_0 n/2 + l^{(s)}]/(2\alpha)$, we present the following summary of properties of $\bar{Q}$ and orthogonal vectors, which will be used later.

Lemma 1. *Under Assumption 1, the following hold.*
(i) The matrix $\bar{Q} \in \mathbb{R}^{r_0 n\times r_0 n}$ has a simple eigenvalue $1-\lambda_1$ with a unit eigenvector η, a simple eigenvalue $1-\lambda_2$ with a unit eigenvector ξ, and an eigenvalue $1-\lambda_3$ with multiplicity $r_0 n - 2$ and with unit eigenvectors $w^{(i)}$, $3 \le i \le r_0 n$. In addition, the vectors η, ξ, $w^{(3)}$, …, $w^{(r_0 n)}$ form an orthonormal basis of $\mathbb{R}^{r_0 n}$.
(ii) If $\{x^{(i)} \in \mathbb{R}^n, 1 \le i \le n\}$ is an orthonormal basis of $\mathbb{R}^n$, then it holds for all $z \in \mathbb{R}^n$ and $1 \le j \le n$ that

$$I_n = \sum_{i=1}^n x^{(i)}(x^{(i)})^T,$$

$$\|z\|^2 = \Big\|\sum_{i=1}^n x^{(i)}(x^{(i)})^T z\Big\|^2 = \sum_{i=1}^n \|x^{(i)}(x^{(i)})^T z\|^2$$

$$= \Big\|\sum_{i=1}^j x^{(i)}(x^{(i)})^T z\Big\|^2 + \Big\|\sum_{i=j+1}^n x^{(i)}(x^{(i)})^T z\Big\|^2.$$

Before presenting main theorems, we provide several properties of the gossip model. The first lemma concerns the explicit expression of the weight sum of the graph $\mathcal{G}$.

Lemma 2. *Under Assumption 1, the weight sum $\alpha = r_0 n[(l_s^{(r)} + l_d^{(r)})r_0 n + 4l^{(s)} - 2l_s^{(r)}]/4$.*

Proof. The conclusion follows directly from the definition of α and Assumption 1. □

The next lemma gives the expression of $\mathbb{E}\{X(t)\}$, and shows how $\mathbb{E}\{X(t)\}$ evolves by decomposing it into three parts, which correspond to the eigenspaces of $\bar{Q}$.

Lemma 3. *Suppose that Assumption 1 holds. Then the expectation of $X(t)$ satisfies that, for all $t \in \mathbb{N}$,*

$$\begin{aligned}
&\mathbb{E}\{X(t)\} \\
&= (1-\lambda_1)^t \eta\eta^T X(0) + \frac{1}{\lambda_1}[1-(1-\lambda_1)^t]\eta\eta^T \bar{R} z^s \\
&+ (1-\lambda_2)^t \xi\xi^T X(0) + \frac{1}{\lambda_2}[1-(1-\lambda_2)^t]\xi\xi^T \bar{R} z^s \\
&+ (1-\lambda_3)^t \sum_{i=3}^{r_0 n} w^{(i)}(w^{(i)})^T X(0) \\
&+ \frac{1}{\lambda_3}[1-(1-\lambda_3)^t] \sum_{i=3}^{r_0 n} w^{(i)}(w^{(i)})^T \bar{R} z^s.
\end{aligned}$$

Proof. Lemma 1 yields that

$$\bar{Q} = (1-\lambda_1)\eta\eta^T + (1-\lambda_2)\xi\xi^T + (1-\lambda_3)\sum_{i=3}^{r_0 n} w^{(i)}(w^{(i)})^T. \tag{4}$$

Then the result follows from (1)–(3). □

We further introduce the following technical assumption.

Assumption 3. Denote $\tilde{l}_+^{(s)} := \max_{1\le j\le s_0 n}\sum_{i\in\mathcal{V}_r} l_{ij}^{(s)}$. It holds that $\tilde{l}_+^{(s)} \le c_l l^{(s)}$ for some constant $c_l > 0$.

Remark 2. The assumption indicates that the maximum influence strength of a stubborn agent on all regular agents is of the same order of $l^{(s)}$. Otherwise, the influence of stubborn agents may not be homogeneous, which could be hard to analyze.

Now we present a lemma that will be used in the proof of main theorems. Denote $X^\eta(t) := \eta\eta^T X(t)$, $X^\xi(t) := \xi\xi^T X(t)$, $\Gamma := \sum_{i=3}^{r_0 n} w^{(i)}(w^{(i)})^T$ (thus, $\Gamma^T\Gamma = \Gamma^2 = \Gamma$), and $X^\Gamma(t) := \Gamma X(t)$. From Lemma 1, $\eta\eta^T + \xi\xi^T + \Gamma = I_{r_0 n}$, and we have the decomposition $X(t) = X^\eta(t) + X^\xi(t) + X^\Gamma(t)$. Further, let

$X^{\perp}(t) := \tilde{\Gamma}X(t)$, where $\tilde{\Gamma} = \xi\xi^T + \Gamma$. We get another decomposition of $X(t)$: $X(t) = X^{\eta}(t) + X^{\perp}(t)$ and $X^{\eta}(t)^T X^{\perp}(t) = 0$. Note that $X^{\eta}(t)+X^{\xi}(t)$ and $X^{\eta}(t)$ represent the average states in each community and everyone's average state, respectively (the ith entry of $X^{\eta}(t) + X^{\xi}(t)$ is $(2\sum_{j\in\mathcal{V}_{rC_i}} X_j(t))/(r_0 n)$). So the two decompositions reveal dynamics of average states. The following lemma gives two upper bounds for the second moment of $X(t)$ under the two decompositions.

Lemma 4. *Suppose that Assumptions 1–3 hold. Denote* $c_s := \sqrt{s_0/r_0}$. *It holds for* $t \in \mathbb{N}$ *and* $n \geq 4/r_0$ *that*

$$\begin{aligned}
&\mathbb{E}\{\|X(t)\|^2\} \\
&\leq (1-\lambda_1)^t\|X^{\eta}(0)\|^2 + (1-\lambda_2)^t\|X^{\xi}(0)\|^2 + (1-\lambda_3)^t\|X^{\Gamma}(0)\|^2 \\
&+ (1\wedge\lambda_1 t)C_{11}c_x^2 r_0 n + \left(\frac{\lambda_2}{\lambda_3}\wedge\lambda_2 t\right)[(1-\lambda_2)^t\|X^{\xi}(0)\|^2 + c_x^2] \\
&+ (1\wedge\lambda_2 t)c_x^2, \qquad (5)\\
&\mathbb{E}\{\|X(t)\|^2\} \\
&\leq (1-\lambda_1)^t\|X^{\eta}(0)\|^2 + (1-\lambda_2\wedge\lambda_3)^t\|X^{\perp}(0)\|^2 \\
&+ (1\wedge\lambda_1 t)C_{21}c_x^2 r_0 n + \left(\frac{\lambda_1}{\lambda_2\wedge\lambda_3}\wedge\lambda_1 t\right)C_{22}c_x^2 r_0 n, \qquad (6)
\end{aligned}$$

where $C_{11} := 3 + c_l/2 + 16c_s c_l + (3c_l + 23)/(2r_0 n)$, $C_{21} := 4c_s c_l + (5 + c_l)/(2r_0 n)$, *and* $C_{22} := 3 + c_l/2 + 4c_s c_l + 2/(r_0 n)$.

Proof. The main idea of the proof is to separately bound the terms $\mathbb{E}\{\|X^{\eta}(t)\|^2\}$, $\mathbb{E}\{\|X^{\xi}(t)\|^2\}$, $\mathbb{E}\{\|X^{\Gamma}(t)\|^2\}$, and $\mathbb{E}\{\|X^{\perp}(t)\|\}^2$. The conclusions then follow from Lemma 1. See Appendix A for the details. □

Remark 3. Lemma 4 provides bounds for how the second moment of agent states evolves over time. Three types of terms appear in the bounds. The first type is an exponentially decreasing term $(1-\lambda_i)^t$, indicating the rate of the averaging update. The second is a linearly increasing term $1\wedge\lambda_i t$, showing cumulative influence of stubborn agents. Lastly, the ratios $\lambda_2 t \wedge (\lambda_2/\lambda_3)$ and $\lambda_1 t \wedge [\lambda_1/(\lambda_2\wedge\lambda_3)]$ indicate the effect of relative influence strength between regular and stubborn agents. The bounds become trivial for $t = \omega(1/\lambda_1)$, and only describe transient behavior.

An immediate consequence of the preceding analysis is that the difference between the agent states and average states can be bounded. For two vectors $X, Y \in \mathbb{R}^{r_0 n}$ and $\varepsilon \in (0,1)$, denote the set of agents i such that $|X_i - Y_i| > \varepsilon c_x$ (i.e., the difference between X_i and Y_i is large) by $\mathcal{S}(X, Y, \varepsilon) := \{i \in \mathcal{V}_r : |X_i - Y_i| > \varepsilon c_x\}$. Then the proof of Lemma 4 ensures the following result.

Corollary 1. *Suppose that Assumptions 1–3 hold. Then it holds for* $t \in \mathbb{N}$, $n \geq 4/r_0$, *and* $\varepsilon, \delta \in (0,1)$ *that*

$$\begin{aligned}
&\mathbb{P}\{|\mathcal{S}(X(t), X^{\eta}(t)+X^{\xi}(t), \varepsilon)| \geq \delta r_0 n\} \\
&\leq \frac{1}{\varepsilon^2\delta}\Big[(1-\lambda_3)^t\frac{\|X^{\Gamma}(0)\|^2}{c_x^2 r_0 n} + \left(\frac{\lambda_1}{\lambda_3}\wedge\lambda_1 t\right)\Big(3 + \frac{c_l}{2} + 10c_s c_l \\
&+ \frac{c_l + 13}{2r_0 n}\Big) + \left(\frac{\lambda_2}{\lambda_3} + \lambda_2 t\right)\left(\frac{(1-\lambda_2)^t\|X^{\xi}(0)\|^2}{c_x^2 r_0 n} + \frac{1}{r_0 n}\right)\Big], \qquad (7)\\
&\mathbb{P}\{|\mathcal{S}(X(t), X^{\eta}(t), \varepsilon)| \geq \delta r_0 n\} \\
&\leq \frac{1}{\varepsilon^2\delta}\Big[(1-\lambda_2\wedge\lambda_3)^t\frac{\|X^{\perp}(t)\|^2}{c_x^2 r_0 n} + \left(\frac{\lambda_1}{\lambda_2\wedge\lambda_3}\wedge\lambda_1 t\right) \\
&\left(3 + \frac{c_l}{2} + 4c_s c_l + \frac{2}{r_0 n}\right)\Big]. \qquad (8)
\end{aligned}$$

Remark 4. The first result (7) bounds the probability of at least $\delta r_0 n$ agents having states at least εc_x away from the average states in their communities. This probability bound decreases first due to the decay of $(1-\lambda_3)^t$ and the relatively small value of $\lambda_i t$, and then increases with t to a constant bound (whether the constant bound is trivial depends on the ratios λ_i/λ_3, $i = 1, 2$, see Corollary 2 for details). The second result (8) shows that there can be a time interval, over which many agents have states close to everyone's average state with high probability.

Proof. To prove the results, note that by the Markov inequality, for $X, Y \in \mathbb{R}^{r_0 n}$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\mathbb{P}\{|\mathcal{S}(X, Y, \varepsilon)| \geq \delta r_0 n\} &\leq \mathbb{P}\{\|X - Y\|^2 \geq \varepsilon^2\delta c_x^2 r_0 n\} \\
&\leq \frac{\mathbb{E}\{\|X - Y\|^2\}}{\varepsilon^2\delta c_x^2 r_0 n}. \qquad (9)
\end{aligned}$$

Now note that $X(t) - X^{\eta}(t) - X^{\xi}(t) = X^{\Gamma}(t)$ and $X(t) - X^{\eta}(t) = X^{\perp}(t)$, so (7) and (8) follow from the upper bounds of $\mathbb{E}\{\|X^{\Gamma}(t)\|^2\}$ and $\mathbb{E}\{\|X^{\perp}(t)\|^2\}$ given in the proof of Lemma 4 ((A.11) and (A.12), respectively). □

The preceding results indicate that we can use average states as references to describe how agent states evolve in finite time. Further analysis based on Lemma 4 can yield stronger results. That is, we can use the expected average states $\mathbb{E}\{X^{\eta}(t) + X^{\xi}(t)\}$ and $\mathbb{E}\{X^{\eta}(t)\}$ as references.

Theorem 1. *Suppose that Assumptions 1–3 hold and* $n \geq 4/r_0$. *Let* $\varepsilon, \delta, \gamma \in (0,1)$ *be such that* $\varepsilon^2\delta\gamma \leq 2/e$.
(i) Assume that

$$\lambda_2 \log[2/(\varepsilon^2\delta\gamma)] < \lambda_3, \qquad (10)$$

$$2\Big[(4 + C_{11})\lambda_1\log\frac{2}{\varepsilon^2\delta\gamma} + C_{12}\Big(1 + \log\frac{2}{\varepsilon^2\delta\gamma}\Big)\lambda_2\Big] < \varepsilon^2\delta\gamma\lambda_3. \qquad (11)$$

Then $(\underline{t}_1, \bar{t}_1) \neq \emptyset$, *and for all* $t \in (\underline{t}_1, \bar{t}_1)$ *it holds that*

$$\mathbb{P}\{|\mathcal{S}(X(t), \mathbb{E}\{X^{\eta}(t) + X^{\xi}(t)\}, \varepsilon)| \geq \delta r_0 n\} \leq \gamma, \qquad (12)$$

where C_{11} *is given in Lemma 4,* $C_{12} = \|X^{\xi}(0)\|^2/(c_x^2 r_0 n) + 1/(r_0 n)$,

$$\underline{t}_1 = \frac{\log[2/(\varepsilon^2\delta\gamma)]}{\lambda_3}, \text{ and } \bar{t}_1 = \frac{\varepsilon^2\delta\gamma/2 - C_{12}\lambda_2/\lambda_3}{(4 + C_{11})\lambda_1 + C_{12}\lambda_2} \wedge \frac{1}{\lambda_2}.$$

(ii) Assume that

$$2\Big[(3 + C_{21})\log\frac{2}{\varepsilon^2\delta\gamma} + C_{22}\Big]\lambda_1 < \varepsilon^2\delta\gamma(\lambda_2\wedge\lambda_3). \qquad (13)$$

Then $(\underline{t}_2, \bar{t}_2) \neq \emptyset$, *and for all* $t \in (\underline{t}_2, \bar{t}_2)$ *it holds that*

$$\mathbb{P}\{|\mathcal{S}(X(t), \mathbb{E}\{X^{\eta}(t)\}, \varepsilon)| \geq \delta r_0 n\} \leq \gamma, \qquad (14)$$

where C_{21} *and* C_{22} *are given in Lemma 4,*

$$\underline{t}_2 = \frac{\log[2/(\varepsilon^2\delta\gamma)]}{\lambda_2\wedge\lambda_3}, \text{ and } \bar{t}_2 = \frac{\varepsilon^2\delta\gamma/2 - C_{22}\lambda_1/(\lambda_2\wedge\lambda_3)}{(3 + C_{21})\lambda_1}.$$

Proof. To prove (12), from (9) it suffices to bound $\mathbb{E}\{\|X(t) - \mathbb{E}\{X^{\eta}(t)\} - \mathbb{E}\{X^{\xi}(t)\}\|^2\}$. From Lemma 1(ii), we have that

$$\begin{aligned}
&\mathbb{E}\{\|X(t) - \mathbb{E}\{X^{\eta}(t)\} - \mathbb{E}\{X^{\xi}(t)\}\|^2\} \\
&= \mathbb{E}\{\|X(t)\|^2\} - 2\mathbb{E}\{X(t)\}^T(\mathbb{E}\{X^{\eta}(t)\} + \mathbb{E}\{X^{\xi}(t)\}) \\
&+ \|\mathbb{E}\{X^{\eta}(t)\} + \mathbb{E}\{X^{\xi}(t)\}\|^2 \\
&= \mathbb{E}\{\|X(t)\|^2\} - \|\mathbb{E}\{X^{\eta}(t)\}\|^2 - \|\mathbb{E}\{X^{\xi}(t)\}\|^2. \qquad (15)
\end{aligned}$$

From Lemma 3 and the Bernoulli inequality, it holds that

$$\begin{aligned}&\|\mathbb{E}\{X^{\eta}(t)\}\|^2\\&=\{(1-\lambda_1)^t\eta^TX(0)+[1-(1-\lambda_1)^t]\zeta_1\}^2\\&\geq(1-\lambda_1)^{2t}\|X^{\eta}(0)\|^2-2\lambda_1t(1-\lambda_1)^t\|X^{\eta}(0)\||\zeta_1|,\\&\|\mathbb{E}\{X^{\xi}(t)\}\|^2\\&=\Big\{(1-\lambda_2)^t\xi^TX(0)+\frac{\lambda_1}{\lambda_2}[1-(1-\lambda_2)^t]\zeta_2\Big\}^2\\&\geq(1-\lambda_2)^{2t}\|X^{\xi}(0)\|^2-2\lambda_1t(1-\lambda_2)^t\|X^{\xi}(0)\||\zeta_2|.\end{aligned}$$

Thus,

$$\begin{aligned}&-\|\mathbb{E}\{X^{\eta}(t)\}\|^2-\|\mathbb{E}\{X^{\xi}(t)\}\|^2\\&\leq-(1-\lambda_1)^{2t}\|X^{\eta}(0)\|^2-(1-\lambda_2)^{2t}\|X^{\xi}(0)\|^2+3\lambda_1tc_x^2r_0n.\end{aligned}$$

Hence, when $1/\lambda_3\leq t\leq1/\lambda_2$, from (5) in Lemma 4, (15) can be bounded by

$$\begin{aligned}&(1-\lambda_3)^t\|X^{\Gamma}(0)\|^2+(4+C_{11})\lambda_1tc_x^2r_0n\\&+\lambda_2\Big(\frac{1}{\lambda_3}+t\Big)[(1-\lambda_2)^t\|X^{\xi}(0)\|^2+c_x^2].\end{aligned}$$

Hence from (9),

$$\begin{aligned}&\mathbb{P}\{|\mathcal{S}(X(t),\mathbb{E}\{X^{\eta}(t)+X^{\xi}(t)\},\varepsilon)|>\delta r_0n\}\\&\leq\frac{1}{\varepsilon^2\delta}\Big[(1-\lambda_3)^t+(4+C_{11})\lambda_1t+C_{12}\Big(\frac{\lambda_2}{\lambda_3}+\lambda_2t\Big)\Big].\end{aligned}$$

When $t\geq\underline{t}_1\geq1/\lambda_3$, $(1-\lambda_3)^t\leq e^{t\log(1-\lambda_3)}\leq e^{-\lambda_3t}\leq e^{-\lambda_3\underline{t}_1}\leq\varepsilon^2\delta\gamma/2$. On the other hand, when $t\leq\bar{t}_1$,

$$(4+C_{11})\lambda_1t+C_{12}\Big(\frac{\lambda_2}{\lambda_3}+\lambda_2t\Big)\leq\frac{\varepsilon^2\delta\gamma}{2}.$$

If the assumptions of (i) hold, then $(\underline{t}_1,\bar{t}_1)\neq\emptyset$ and the conclusion follows. The second part of the theorem can be derived similarly from (6). □

Remark 5. The results provide bounds for the probability of agent states concentrating around expected average states. For the case where stubborn agents have small influence and the influence strength within communities is much larger than that between communities (λ_1 and λ_2 much smaller than λ_3), (12) indicates that most agent states concentrate around expected average states within communities and form transient clusters. If the stubborn-agent influence is small but the influence strength within and between communities is similar (λ_1 much smaller than $\lambda_2\wedge\lambda_3$, but λ_2 and λ_3 are similar), (14) indicates that most agent states concentrate around everyone's expected average state. See Corollary 2 for detailed discussion on how link strength parameters influence the concentration. To obtain a nontrivial bound, δ (the proportion of regular agents whose states do not concentrate) and γ (the probability that the concentration does not occur) need to be small. But ε (the error of the concentration) does not need. For example, it is sufficient to set ε to be less than half of the distance between average states of the two communities, so that the two transient clusters can be distinguished. For a fixed network, setting smaller ε and δ leads to a large bound of γ, as stronger concentration occurs with less probability. Since λ_1, λ_2, and λ_3 depend on the network size n, n has to be large enough to ensure that (11) and (13) hold with small γ. Simulation (Figs. 2 and 3) illustrates that such concentration can occur over small networks. Studying sharp bounds for the concentration probability is left to future work.

Remark 6. Here we compare the obtained results with existing transient behavior analysis. Dietrich et al. (2016) study a generalized HK model and define a transient cluster as a subgroup of agents whose opinion range decreases faster than the distance of the subgroup from other agents. We characterize agent states based on their distance from average states. Such a framework includes situations where clusters approach each other, but defining such references requires specifying subgroups beforehand. Shree et al. (2022) provide bounds for opinion difference in finite time for a stochastic bounded confidence model. These bounds coincide with the asymptotic behavior. In contrast, we study transient behavior of the gossip model that is different from the asymptotic behavior. For the gossip model without stubborn agents, Fagnani and Zampieri (2008) show that $\|X(t)-X^{\eta}(t)\|^2$ is close to its expectation at the early stage of the process, which provides insight into why both average states and expected average states can be used as references. We will study such concentration in the future.

It is possible to derive bounds similar to Theorem 1 for $\mathcal{S}(X(t),X^{\eta}(0)+X^{\xi}(0),\varepsilon)$ and $\mathcal{S}(X(t),X^{\eta}(0),\varepsilon)$, i.e., the deviation of agent states from average initial states. We state the results in the next theorem.

Theorem 2. *Suppose that Assumptions 1–3 hold and $n\geq4/r_0$. Let $\varepsilon,\delta,\gamma\in(0,1)$ be such that $\varepsilon^2\delta\gamma\leq2/e$.*
(i) If (10) and (11) hold, then we have that for all $t\in(\underline{t}_1,\bar{t}_1)\neq\emptyset$, where $\underline{t}_1$ and $\bar{t}_1$ are given in Theorem 1(i),

$$\mathbb{P}\{|\mathcal{S}(X(t),X^{\eta}(0)+X^{\xi}(0),\varepsilon)|\geq\delta r_0n\}\leq\gamma.\tag{16}$$

(ii) If (13) holds, then we have that for all $t\in(\underline{t}_2,\bar{t}_2)\neq\emptyset$, where $\underline{t}_2$ and $\bar{t}_2$ are given in Theorem 1(ii),

$$\mathbb{P}\{|\mathcal{S}(X(t),X^{\eta}(0),\varepsilon)|\geq\delta r_0n\}\leq\gamma.\tag{17}$$

Proof. The proof is similar to that of Theorem 1. See Appendix B for the details. □

Remark 7. Theorems 1 and 2 provide similar probability bounds for concentration of agent states around expected average states and average initial states. However, concentration around expected average states may require a much smaller network size than the latter, as shown in Section 4. Future work will explore sharp bounds to distinguish between the two types of concentration.

So far we have studied concentration around average states under conditions of λ_i. As a consequence of Theorem 2, the following corollary shows how such phenomena depend on link strength parameters $l_s^{(r)}$, $l_d^{(r)}$, and $l^{(s)}$. Parallel results hold for the expected average states.

Corollary 2. *Suppose that Assumptions 1–3 hold and denote $l_0^{(s)}:=l^{(s)}/(r_0n)$.*
(i) If $l_s^{(r)}=\omega(l_d^{(r)}\vee l_0^{(s)})$, then for large enough n depending on $l_s^{(r)}$, $l_d^{(r)}$, and $l^{(s)}$, (16) holds for all $t\in(\underline{t}_1,\bar{t}_1)$ with

$$\underline{t}_1=\Big(\Theta(1)+O\Big(\frac{l_0^{(s)}}{l_s^{(r)}}\Big)\Big)r_0n,\ \text{and}\ \bar{t}_1=\Theta\Big(\frac{l_s^{(r)}}{l_d^{(r)}\vee l_0^{(s)}}\Big)r_0n.$$

(ii) If $l_s^{(r)}\leq l_d^{(r)}+o(l_d^{(r)})$ and $l_d^{(r)}=\omega(l_0^{(s)})$, then for large enough n depending on $l_d^{(r)}$ and $l^{(s)}$, (17) holds for all $t\in(\underline{t}_2,\bar{t}_2)$ with

$$\underline{t}_2=\Big(\Theta(1)+O\Big(\frac{[0\vee(l_s^{(r)}-l_d^{(r)})]+l_0^{(s)}}{l_d^{(r)}}\Big)\Big)r_0n,\ \text{and}$$

$$\bar{t}_2=\Theta\Big(\frac{l_d^{(r)}}{l_0^{(s)}}\Big)r_0n.$$

Table 1
Summary of main results.

Conditions	Behavior of agent states	Results
$l_s^{(r)} = \omega(l_d^{(r)} \wedge l_0^{(s)})$ with $l_0^{(s)} = l^{(s)}/(r_0 n)$ (stubborn influence is small, intra-community influence is large)	Concentrate around average states/expected average states/average initial states within communities over the time interval $(\Theta(r_0 n), \Theta(l_s^{(r)} r_0 n/(l_d^{(r)} \vee l_0^{(s)})))$	(7), (12), Theorem 2(i), Corollary 2(i)
$l_s^{(r)} \le l_d^{(r)} + o(l_d^{(r)})$ and $l_d^{(r)} = \omega(l_0^{(s)})$ (stubborn influence is small, intra-community influence is moderate or small)	Concentrate around everyone's average state/expected average state/average initial state over the time interval $(\Theta(r_0 n), \Theta(l_d^{(r)} r_0 n/l_0^{(s)}))$	(8), (14), Theorem 2(ii), Corollary 2(ii)
$l_s^{(r)} \vee l_d^{(r)} = o(l_0^{(s)})$ (stubborn influence is large)	Close to the stationary distribution over the time interval $(\Theta(r_0 n \log(r_0 n)), +\infty)$	Corollary 3

Proof. From the definition of λ_i, it follows that (10) and (11) are equivalent to

$$2\Big(\log\frac{2}{\varepsilon^2\delta\gamma}-1\Big)l_0^{(s)} + \Big(2\log\frac{2}{\varepsilon^2\delta\gamma}-1\Big)l_d^{(r)} < l_s^{(r)}, \tag{18}$$

$$2\Big[2(4+C_{11}+C_{12})\log\frac{2}{\varepsilon^2\delta\gamma} + 2C_{12} - \varepsilon^2\delta\gamma\Big]l_0^{(s)} + \Big[4C_{12}\Big(1+\log\frac{2}{\varepsilon^2\delta\gamma}\Big) - \varepsilon^2\delta\gamma\Big]l_d^{(r)} < \varepsilon^2\delta\gamma l_s^{(r)}. \tag{19}$$

The condition $l_s^{(r)} = \omega(l_d^{(r)} \vee l_0^{(s)})$ guarantees that (18) and (19) hold for large enough n that depends on $l_s^{(r)}$, $l_d^{(r)}$, and $l^{(s)}$. Hence the conclusion follows from the expression of $\underline{t}_1$ and $\bar{t}_1$ given in Theorem 1. The proof of (ii) is similar. It suffices to note that (13) is equivalent to

$$2\Big[2(3+C_{21})\log\frac{2}{\varepsilon^2\delta\gamma} + 2C_{22} - \varepsilon^2\delta\gamma\Big]l_0^{(s)} < \varepsilon^2\delta\gamma[l_d^{(r)} + (l_d^{(r)} \wedge l_s^{(r)})]. \tag{20}$$

The condition $l_d^{(r)} = \omega(l_0^{(s)})$ ensures that (20) holds with large enough n that depends on $l_d^{(r)}$ and $l^{(s)}$. □

Remark 8. The first part of the corollary indicates that, if the intra-community weights are of higher order than the inter-community weights and the average weight between regular and stubborn agents, most agent states concentrate around the initial average opinion of the corresponding community over a finite time interval. The length of this interval depends on relative strength within communities to that between communities and between regular and stubborn agents. The assumption $l_s^{(r)} \le l_d^{(r)} + o(l_d^{(r)})$ of the second part means that intra-community weights are less than or slightly larger than inter-community weights. If stubborn agents have small influence, then most agent states are close to everyone's initial average opinion. Since $l_s^{(r)} \le l_d^{(r)} + o(l_d^{(r)})$ implies $l_s^{(r)} = O(l_d^{(r)})$, the corollary characterizes a phase transition phenomenon.

The preceding results study the case where the influence of stubborn agents is small. Now we investigate the case with large stubborn influence. The gossip model (1) converges to a unique stationary distribution π (i.e., if $X(0)$ has distribution π, then $X(t)$ has the same distribution for all $t \in \mathbb{N}^+$), if there is at least one stubborn agent and the network is connected (Acemoğlu et al., 2013; Xing et al., 2023). To measure the distance between transient and stationary distributions, we introduce the Wasserstein metric between two measures μ and ν,

$$d_W(\mu,\nu) := \inf_{(X,Y)\in J} \mathbb{E}\{\|X-Y\|\},$$

where J is the set of random vector pairs (X, Y) such that the marginal distributions of X and Y are μ and ν, respectively. The following theorem provides the time interval over which the distribution of $X(t)$ is close to π.

Theorem 3. *Suppose that Assumptions 1–2 hold and $l^{(s)} > 0$. Then, for $\varepsilon \in (0, 1)$, it holds that*

$$d_W(X(t),\pi) \le \varepsilon, \ \ \forall t > \frac{\log\{c_x(r_0 n)^{\frac{5}{2}}[1+1/(2\lambda_1)]/\varepsilon\}}{\log[1/(1-\lambda_1)]}.$$

Proof. The conclusion is obtained from a coupling argument. See Appendix C. □

Theorem 3 characterizes asymptotic behavior of the gossip model. The next corollary focuses on the case where the influence of stubborn agents is large.

Corollary 3. *Suppose that Assumptions 1–2 hold. If $(l_d^{(r)} \vee l_s^{(r)})r_0 n = o(l^{(s)})$, then $d_W(X(t),\pi) \le \varepsilon$ holds for $\varepsilon \in (0, 1)$ and $t > t_0$ with $t_0 = t_0(\varepsilon) = \Theta(r_0 n \log(r_0 n))$.*

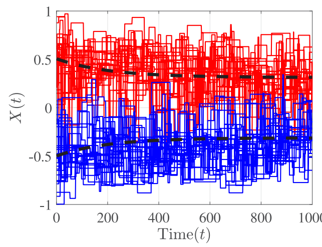
Remark 9. Corollary 3 indicates that, if the influence of stubborn agents is large, then the distribution of $X(t)$ can be close to the stationary distribution of the gossip model at the early stage of the process.

3.2. Discussion and extension

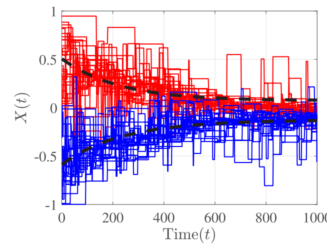
In this subsection, we first summarize obtained transient behavior under different parameter settings, and then discuss the extensions of the results.

In the previous subsection, we obtained several probability bounds for agent states concentrating around average states (Corollary 1), expected average states (Theorem 1), and average initial states (Theorem 2). Note that these bounds are the same except for some constants. So the explicit dependence of transient behavior on link strength parameters $l_s^{(r)}$, $l_d^{(r)}$, and $l^{(s)}$, given in Corollary 2, still holds for average states and expected average states. Table 1 summarizes the findings. The results indicate a phase transition phenomenon: the model behaves differently at the early stage of the process under different parameter settings.

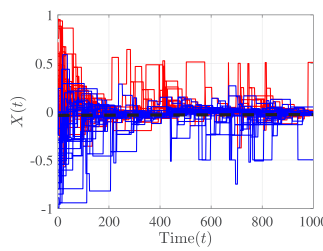
Note that the obtained bounds are not tight. As shown in Section 4, expected average states are good references for transients. The concentration occurs for small networks. In contrast, concentration around average initial states requires much larger n. The key idea of studying transient behavior is to find references such as the expected average states and to bound the deviation of agent states from such references, so it is possible to generalize the obtained results to multiple-community cases. Matrix perturbation theory can be used to analyze the case involving unequal-sized communities or heterogeneous influence of stubborn agents (i.e., Assumption 1(iii) does not hold). Studying the gossip model over an SBM needs analysis of the deviation of the random graph from its expectation based on concentration inequalities (e.g. Chung and Radcliffe (2011)).



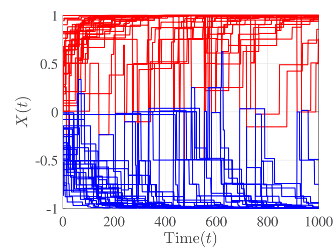
(a) The case where $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log n)/n$, and $l^{(s)} = (s_0 \log^2 n)/2$. Most agent states in the same community are close to the expected average state of that community (black dashed lines).



(b) The case where $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log n)/n$, and $l^{(s)} = (s_0 \log n)/2$. Agents behave similar to Fig. 2(a), but move towards the origin in the long run.



(c) The case where $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log^{2.4} n)/n$, and $l^{(s)} = (s_0 \log n)/2$. Most states are close to everyone's expected average state (the black dashed line), and keep fluctuating.



(d) The case where $l_s^{(r)} = l_d^{(r)} = (\log n)/n$ and $l^{(s)} = (s_0 \log^3 n)/2$. Most agents move quickly towards stubborn agents during the initial phase, and stay polarized.

Fig. 2. Behavior of the gossip model. The figure illustrates the evolution of regular-agent states under four sets of parameters. The color of a trajectory represents the community label of that agent (red represents $\mathcal{V}_{r1}$, and blue represents $\mathcal{V}_{r2}$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

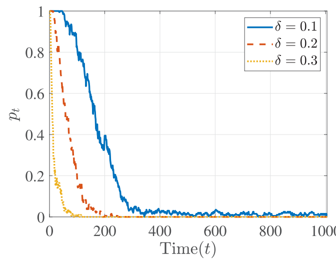
4. Numerical simulation

In this section, we conduct numerical experiments to validate the theoretical results obtained in Section 3.

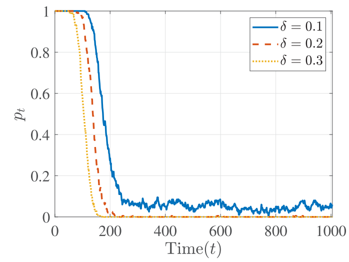
In Fig. 2 we demonstrate transient behavior of the gossip model. Set the network size $n = 60$, the bound of agent states $c_x = 1$, and the proportion of regular agents $r_0 = 0.9$. Generate the initial value $X_i(0)$ independently from uniform distribution on $(0, 1)$ for all $i \in \mathcal{V}_{r1}$, $X_j(0)$ independently from uniform distribution on $(-1, 0)$ for all $j \in \mathcal{V}_{r2}$, and set stubborn-agent states $z^s = [\mathbf{1}^T_{s_0 n/2} \ -\mathbf{1}^T_{s_0 n/2}]^T$. First, we set the edge weight between agents in the same community to be $l_s^{(r)} = (\log^{2.5} n)/n$, greater than the edge weight between communities $l_d^{(r)} = (\log n)/n$. The edge weights between regular and stubborn agents are as follows: for all $i \in \mathcal{V}_{r1}$, $l_{ij}^{(s)} \equiv (\log^2 n)/n$ for $1 \le j \le s_0 n/2$, and $l_{ij}^{(s)} \equiv 0$ for $1 + s_0 n/2 \le j \le s_0 n$; for all $i \in \mathcal{V}_{r2}$, $l_{ij}^{(s)} \equiv 0$ for $1 \le j \le s_0 n/2$, and $l_{ij}^{(s)} \equiv (\log^2 n)/n$ for $1 + s_0 n/2 \le j \le s_0 n$. Hence $l^{(s)} = (s_0 \log^2 n)/2$. This setting intuitively means that the first half of stubborn agents influence only $\mathcal{V}_{r1}$, whereas the second half influence only $\mathcal{V}_{r2}$. The setting is used in all numerical experiments. Fig. 2(a) shows that agents form two transient clusters corresponding to their community labels and centered around the expected average states within communities. Asymptotic behavior of the system can be different from what is shown in Fig. 2(a). As an example, set $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log n)/n$, and $l^{(s)} = (s_0 \log n)/2$. In Fig. 2(b), agents in the same community also form transient clusters, but they get close in the end. Next, we set $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log^{2.4} n)/n$, and $l^{(s)} = (s_0 \log n)/2$. That is, the influence of stubborn agents is small, and inter-community influence strength is similar to intra-community strength. Hence all agents form a single transient cluster and concentrate around everyone's average state, as demonstrated in Fig. 2(c). However, they cannot reach a consensus and keep fluctuating, due to the presence of stubborn agents. Finally, set $l_s^{(r)} = l_d^{(r)} = (\log n)/n$ and $l^{(s)} = (s_0 \log^3 n)/2$. Fig. 2(d) shows that agent states move quickly towards the positions of stubborn agents if the latter have large influence.

Next we study the probability of concentration around average states, investigated in Theorems 1 and 2. We first study concentration around expected average states. Set $n = 60$, $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log n)/n$, and $l^{(s)} = (s_0 \log n)/2$ (i.e., intra-community edge weights are large). The model is run for 200 times and the final time step is $\lfloor n(\log n)^2 \rfloor$. We estimate the probability p_t that the concentration fails by computing $\frac{1}{200}\sum_{k=1}^{200} \mathbb{I}_{[|\mathcal{S}_t^{\varepsilon}(k)| \ge \delta r_0 n]}$, where $\mathbb{I}_{[\cdot]}$ is the indicator function. Here $\mathcal{S}_t^{\varepsilon}(k)$ is the set $\mathcal{S}(X(t), \mathbb{E}\{X^{\eta}(t) + X^{\xi}(t)\}, \varepsilon)$ at the kth run, that is, the set of agents with states at least εc_x away from the expected average state of their communities at time t. The concentration error ε and the proportion of non-concentrating agents δ are defined in Theorem 1. Fig. 3(a) shows that p_t first decreases with time and then fluctuates around a constant, and it decreases with δ, validating Theorem 1(i). Although the theoretical bound is not tight (e.g., the ratio $\lambda_2/(\lambda_3 \varepsilon^2 \delta) \approx 18.7 > 1$), the experiment indicates that the concentration still occurs for small networks. Now we set $l_d^{(r)} = (\log^{2.4} n)/n$ for the case of moderate intra-community influence. Fig. 3(b) illustrates that the probability varies similarly, aligning with Theorem 1(ii). Next we examine the concentration around average initial states. Fig. 3(c) shows that the concentration around average initial states within communities happens with less probability than that around expected average states (presented in Fig. 3(a)). Larger networks ensure more concentration (Fig. 3(d) with $n = 500$). The concentration around everyone's average initial state, given in Fig. 3(e), is similar to Fig. 3(b).

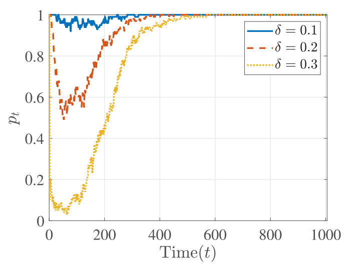
Finally, we conduct two experiments to demonstrate how concentration around average initial states changes when edge weights vary. First, we set $l_d^{(r)} = (\log^{\beta_2} n)/n = (\log^5 n)/n$ and $l^{(s)} = \log n$, and consider $l_s^{(r)} = (\log^{\beta_1} n)/n$ with $\beta_1 = 1, \ldots, 10$. We run the gossip model for 50 times, for each value of β_1 and



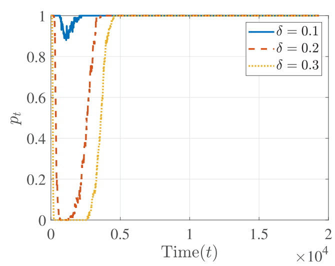
(a) Concentration around expected average states within communities, where $n = 60$, $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log n)/n$, and $\varepsilon = 0.3$.



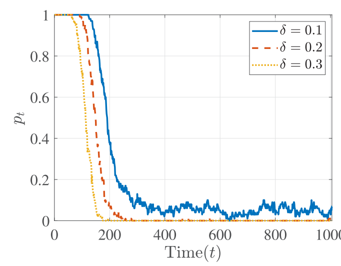
(b) Concentration around everyone's expected average state, where $n = 60$, $l_s^{(r)} = (\log^{2.5} n)/n$, $l_d^{(r)} = (\log^{2.4} n)/n$, and $\varepsilon = 0.2$.



(c) Concentration around average initial states within communities, where n, $l_s^{(r)}$, $l_d^{(r)}$, and ε are the same as (a).

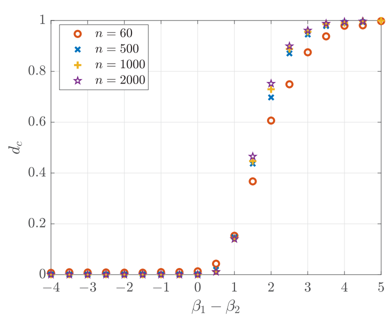


(d) Concentration around average initial states within communities, where $l_s^{(r)}$, $l_d^{(r)}$, and ε are the same as (a) and $n = 500$.

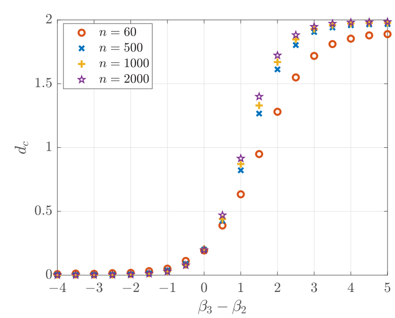


(e) Concentration around everyone's average initial state, where n, $l_s^{(r)}$, $l_d^{(r)}$, and ε are the same as (b).

Fig. 3. The probability p_t of at least δ proportion of agents with states at least εc_x away from average states at time t. In all experiments, $l^{(s)} = (s_0 \log n)/2$. The expected average states are considered in (a–b), whereas the average initial states in (c–e).



(a) The case where $l_s^{(r)}$ varies. $\beta_2 = 5$ and $\beta_3 = 1$.



(b) The case where $l^{(s)}$ varies. $\beta_1 = 1$ and $\beta_2 = 5$.

Fig. 4. Phase transition of transient behavior of the gossip model. In the experiments, we set $l_s^{(r)} = (\log^{\beta_1} n)/n$, $l_d^{(r)} = (\log^{\beta_2} n)/n$, and $l^{(s)} = \log^{\beta_3} n$, and compute $d_c = 2|\sum_{i\in\mathcal{V}_{r1}} X_i(t) - \sum_{i\in\mathcal{V}_{r2}} X_i(t)|/(r_0 n)$ with $t = \lfloor n \log n \rfloor$.

for $n = 60, 500, 1000, 2000$, and compute the difference $d_c = 2|\sum_{i\in\mathcal{V}_{r1}} X_i(t) - \sum_{i\in\mathcal{V}_{r2}} X_i(t)|/(r_0 n)$ with $t = \lfloor n \log n \rfloor$. Thus d_c represents the difference between the averages of agent states in the two communities. The case $d_c = 1$ means local concentration whereas $d_c = 0$ represents global concentration. Fig. 4(a) presents the phase transition phenomenon (the effect becomes stronger as n grows). It shows that the local concentration appears when $\beta_1 - \beta_2 > 0$ (i.e., $l_d^{(r)} = o(l_s^{(r)})$), as predicted by Corollary 2. On the other hand, Fig. 4(a) indicates that the global concentration appears when $\beta_1 - \beta_2 < 0$. In the second experiment, we set $l_s^{(r)} = (\log n)/n$ and $l_d^{(r)} = (\log^{\beta_2} n)/n = (\log^5 n)/n$, and consider $l^{(s)} = \log^{\beta_3} n$ with $\beta_3 = 1, \ldots, 10$. This examines the phase transition in the influence of stubborn agents. Hence, $d_c = 2$ means regular agents have states close to stubborn ones, and $d_c = 0$ represents the global concentration. Fig. 4(b) shows that the transition occurs at $nl_d^{(s)} = \Theta(l^{(s)})$ (i.e., $\beta_3 = \beta_2$), validating Corollary 3.

5. Conclusion

In this paper, we investigated transient behavior of the gossip model with two communities. By analyzing the second moment of agent states, we found a phase transition phenomenon: When edge weights within communities are large and weights between regular and stubborn agents are small, most agents have states close to the average opinion within their communities at the early stage of the process. When the difference between intra- and inter-community weights is small, most agents have states close to everyone's average opinion. In contrast, if weights between regular and stubborn agents are large, the distribution of agent states is close to the stationary distribution. Future work includes to study the gossip and other models over general graphs, and to link theoretical findings to empirical data.

Acknowledgments

The authors would like to thank the anonymous reviewers for their insightful comments and suggestions.

Appendix A. Proof of Lemma 4

Proof of (5). We divide the proof into four steps. Step 1 provides upper and lower bounds for $\mathbb{E}\{\|X^\eta(t)\|^2\}$. Bounds of $\mathbb{E}\{\|X^\xi(t)\|^2\}$ are given in step 2. The bounds are used in step 3, obtaining an upper bound for $\mathbb{E}\{\|X^\Gamma(t)\|^2\}$. Step 4 concludes the proof.

Step 1 (Bounding $\mathbb{E}\{\|X^\eta(t)\|^2\}$). Note that

$$\begin{aligned}&\mathbb{E}\{\|X^\eta(t+1)\|^2|X(t)\}\\&=\mathbb{E}\{X(t+1)^T\eta\eta^T\eta\eta^TX(t+1)|X(t)\}\\&=\mathbb{E}\{X(t)^TQ(t)^T\eta\eta^TQ(t)X(t)|X(t)\}+2\mathbb{E}\{X(t)^TQ(t)^T\\&\eta\eta^TR(t)z^s|X(t)\}+\mathbb{E}\{(z^s)^TR(t)^T\eta\eta^TR(t)z^s|X(t)\}\\&=X(t)^T\mathbb{E}\{Q(t)^T\eta\eta^TQ(t)\}X(t)+2X(t)^T\mathbb{E}\{Q(t)^T\eta\eta^TR(t)\}z^s\\&+(z^s)^T\mathbb{E}\{R(t)^T\eta\eta^TR(t)\}z^s. \qquad (A.1)\end{aligned}$$

In the last equation, $X(t)$ and z^s are taken out of the conditional expectations due to measurability, and the conditional expectations degenerate into expectations because $[Q(t), R(t)]$ is independent of $X(t)$. Let $\mathcal{E}_r$ be the collection of edges whose end points are regular agents, and let $\mathcal{E}_s$ be the set of edges connecting regular and stubborn agents. Hence, $\mathcal{E}=\mathcal{E}_r\cup\mathcal{E}_s$. The definition of $Q(t)$ implies that $Q(t)^T\eta\eta^TQ(t)=\eta\eta^T$ when an edge in $\mathcal{E}_r$ is selected, and $Q(t)^T\eta\eta^TQ(t)=(\eta-\eta_ue_u/2)(\eta-\eta_ue_u/2)^T$ if $\{u,v\}\in\mathcal{E}_s$ is selected. So

$$\begin{aligned}&\mathbb{E}\{Q(t)\eta\eta^TQ(t)\}\\&=\eta\eta^T-\frac{l^{(s)}}{2\alpha}\sum_{u\in\mathcal{V}_r}\eta_ue_u\eta^T-\frac{l^{(s)}}{2\alpha}\sum_{u\in\mathcal{V}_r}\eta(\eta_ue_u)^T+\frac{l^{(s)}}{4\alpha}\sum_{u\in\mathcal{V}_r}\eta_u^2e_ue_u^T\\&=(1-2\lambda_1)\eta\eta^T+\frac{\lambda_1}{2r_0n}I_{r_0n},\end{aligned}$$

where $e_u:=e_u^{(r_0n)}$. Denote $e_v^s:=e_v^{(s_0n)}$, $\tilde{l}_v^{(s)}:=\sum_{u\in\mathcal{V}_r}l_{uv}^{(s)}$, and $\tilde{\eta}:=\sum_{1\le v\le s_0n}\tilde{l}_v^{(s)}e_v^s/\sqrt{s_0n}$. Then

$$\begin{aligned}&\mathbb{E}\{Q(t)^T\eta\eta^TR(t)\}\\&=\sum_{\{u,v+r_0n\}\in\mathcal{E}_s}\left(\eta-\frac{1}{2}\eta_ue_u\right)\left(\frac{1}{2}\eta_ue_v^s\right)^T\frac{l_{uv}^{(s)}}{\alpha}\\&=\sum_{1\le v\le s_0n}\sum_{u\in\mathcal{V}_r}\left(\frac{l_{uv}^{(s)}}{2\alpha}\eta_u\eta(e_v^s)^T-\frac{l_{uv}^{(s)}}{4\alpha}\eta_u^2e_u(e_v^s)^T\right)\\&=\frac{c_s\lambda_1}{l^{(s)}}\eta\tilde{\eta}^T-\frac{\lambda_1}{2r_0nl^{(s)}}\tilde{M},\\&\mathbb{E}\{R(t)^T\eta\eta^TR(t)\}\\&=\frac{1}{4}\sum_{1\le v\le s_0n}\sum_{u\in\mathcal{V}_r}\frac{l_{uv}^{(s)}}{\alpha}\eta_u^2e_v^s(e_v^s)^T=\frac{\lambda_1\tilde{D}^{(s)}}{2r_0nl^{(s)}},\end{aligned}$$

where $\tilde{D}^{(s)}:=\sum_{1\le v\le s_0n}\tilde{l}_v^{(s)}e_v^s(e_v^s)^T$. Hence, (A.1) is

$$\begin{aligned}&X(t)^T\left[(1-2\lambda_1)\eta\eta^T+\frac{\lambda_1}{2r_0n}I_{r_0n}\right]X(t)+2X(t)^T\\&\left[\frac{c_s\lambda_1}{l^{(s)}}\eta\tilde{\eta}^T-\frac{\lambda_1}{2r_0nl^{(s)}}\tilde{M}\right]z^s+(z^s)^T\frac{\lambda_1\tilde{D}^{(s)}}{2r_0nl^{(s)}}z^s\\&=(1-2\lambda_1)\|X^\eta(t)\|^2+\frac{\lambda_1}{2r_0n}\|X(t)\|^2+\frac{2c_s\lambda_1}{l^{(s)}}X(t)^T\eta\tilde{\eta}^Tz^s\\&-\frac{\lambda_1}{r_0nl^{(s)}}X(t)^T\tilde{M}z^s+\frac{\lambda_1}{2r_0nl^{(s)}}(z^s)^T\tilde{D}^{(s)}z^s.\end{aligned}$$

Taking expectation yields the following upper and lower bounds for $\mathbb{E}\{\|X^\eta(t)\|^2\}$ when $n\ge 1/r_0$,

$$\begin{aligned}&\mathbb{E}\{\|X^\eta(t+1)\|^2\}\\&\le\left(1-\frac{3}{2}\lambda_1\right)\mathbb{E}\{\|X^\eta(t)\|^2\}+\frac{\lambda_1}{2r_0n}(\mathbb{E}\{\|X^\xi(t)\|^2\}+\mathbb{E}\{\|X^\Gamma(t)\|^2\})\\&+\frac{2c_s\lambda_1}{l^{(s)}}\mathbb{E}\{X(t)\}^T\eta\tilde{\eta}^Tz^s-\frac{\lambda_1}{r_0nl^{(s)}}\mathbb{E}\{X(t)\}^T\tilde{M}z^s+\frac{\lambda_1\tilde{l}_+^{(s)}}{2r_0nl^{(s)}}\|z^s\|^2, \qquad (A.2)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\{\|X^\eta(t+1)\|^2\}&\ge(1-2\lambda_1)\mathbb{E}\{\|X^\eta(t)\|^2\}+\frac{2c_s\lambda_1}{l^{(s)}}\mathbb{E}\{X(t)\}^T\eta\tilde{\eta}^Tz^s\\&-\frac{\lambda_1}{r_0nl^{(s)}}\mathbb{E}\{X(t)\}^T\tilde{M}z^s. \qquad (A.3)\end{aligned}$$

Furthermore, by induction and from (A.2), Assumptions 2 and 3, and Lemma 5 at the end of this section,

$$\begin{aligned}&\mathbb{E}\{\|X^\eta(t)\|^2\}\le(1-\lambda_1)^t\|X^\eta(0)\|^2+\lambda_1\left(\frac{1}{\lambda_1}\wedge t\right)\\&\left[2c_sc_l(\|X^\eta(0)\|+|\zeta_1|)\|z^s\|+\frac{1}{2r_0n}(c_x^2r_0n+\|X(0)\|^2\right.\\&\left.+3\|\tilde{z}^s\|^2+c_l\|z^s\|^2)\right], \qquad (A.4)\end{aligned}$$

where $\tilde{z}^s:=\tilde{M}z^s/l^{(s)}$ and $\zeta_1:=\eta^T\bar{R}z^s/\lambda_1=\eta^T\tilde{M}z^s/l^{(s)}$.

Step 2 (Bounding $\mathbb{E}\{\|X^\xi(t)\|^2\}$). Note that

$$\begin{aligned}&\mathbb{E}\{\|X^\xi(t+1)\|^2|X(t)\}\\&=\mathbb{E}\{X(t)^TQ(t)^T\xi\xi^TQ(t)X(t)|X(t)\}+2\mathbb{E}\{X(t)^TQ(t)^T\\&\xi\xi^TR(t)z^s|X(t)\}+\mathbb{E}\{(z^s)^TR(t)^T\xi\xi^TR(t)z^s|X(t)\}\\&=X(t)^T\mathbb{E}\{Q(t)^T\xi\xi^TQ(t)\}X(t)+2X(t)^T\mathbb{E}\{Q(t)^T\xi\xi^TR(t)\}z^s\\&+(z^s)^T\mathbb{E}\{R(t)^T\xi\xi^TR(t)\}z^s, \qquad (A.5)\end{aligned}$$

where the last equation is obtained in the same way as (A.1). Denote $\mathcal{E}_r^{=}:=\{\{u,v\}\in\mathcal{E}_r:\mathcal{C}_u=\mathcal{C}_v\}$ and $\mathcal{E}_r^{\neq}:=\{\{u,v\}\in\mathcal{E}_r:\mathcal{C}_u\neq\mathcal{C}_v\}$. For $Q(t)^T\xi\xi^TQ(t)$, we have that

$$\begin{aligned}&\mathbb{E}\{Q(t)^T\xi\xi^TQ(t)\}\\&=\sum_{\{u,v\}\in\mathcal{E}_r^{=}}\frac{l_s^{(r)}}{\alpha}\xi\xi^T+\sum_{\{u,v\}\in\mathcal{E}_r^{\neq}}\frac{l_d^{(r)}}{\alpha}(\xi-\xi_ue_u-\xi_ve_v)\\&(\xi-\xi_ue_u-\xi_ve_v)^T+\sum_{\{u,v\}\in\mathcal{E}_s}\frac{l_{u,v-r_0n}^{(s)}}{\alpha}\left(\xi-\frac{1}{2}\xi_ue_u\right)\left(\xi-\frac{1}{2}\xi_ue_u\right)^T\\&=\left(\sum_{\{u,v\}\in\mathcal{E}_r^{=}}+\sum_{\{u,v\}\in\mathcal{E}_r^{\neq}}+\sum_{\{u,v\}\in\mathcal{E}_s}\right)\left(\frac{a_{uv}}{\alpha}\xi\xi^T\right)\\&+\frac{l_d^{(r)}}{\alpha}\sum_{\{u,v\}\in\mathcal{E}_r^{\neq}}[-\xi(\xi_ue_u+\xi_ve_v)^T-(\xi_ue_u+\xi_ve_v)\xi^T]\\&+\frac{l_d^{(r)}}{\alpha}\sum_{\{u,v\}\in\mathcal{E}_r^{\neq}}(\xi_ue_u+\xi_ve_v)(\xi_ue_u+\xi_ve_v)^T\\&+\sum_{u\in\mathcal{V}_r}\sum_{1\le v\le s_0n}\frac{l_{uv}^{(s)}}{2\alpha}(-\xi_u\xi e_u^T-\xi_ue_u\xi^T)+\sum_{u\in\mathcal{V}_r}\sum_{1\le v\le s_0n}\frac{l_{uv}^{(s)}}{4\alpha}\xi_u^2e_ue_u^T\\&=\xi\xi^T-\frac{l_d^{(r)}}{\alpha}r_0n\xi\xi^T+\frac{l_d^{(r)}}{\alpha}\sum_{\{u,v\}\in\mathcal{E}_r^{\neq}}(\xi_ue_u+\xi_ve_v)(\xi_ue_u+\xi_ve_v)^T\\&-\frac{l^{(s)}}{\alpha}\xi\xi^T+\frac{l^{(s)}}{4\alpha r_0n}I_{r_0n}\\&=\left(1-\frac{l_d^{(r)}r_0n+l^{(s)}}{\alpha}\right)\xi\xi^T+\frac{\lambda_1}{2r_0n}I_{r_0n}\\&+\frac{2(\lambda_2-\lambda_1)}{r_0n}\sum_{\{u,v\}\in\mathcal{E}_r^{\neq}}(\xi_ue_u+\xi_ve_v)(\xi_ue_u+\xi_ve_v)^T\end{aligned}$$

$$= (1-2\lambda_2)\xi\xi^T + \frac{\lambda_1}{2r_0 n} I_{r_0 n} + \frac{2(\lambda_2-\lambda_1)}{r_0 n}\Big(\xi\xi^T + \frac{1}{2}\Gamma\Big),$$

where the last equation is obtained from

$$\sum_{\{u,v\}\in\mathcal{E}_r^{\neq}} (\xi_u e_u + \xi_v e_v)(\xi_u e_u + \xi_v e_v)^T$$
$$= \frac{1}{2} I_{r_0 n} + \sum_{u\in\mathcal{V}_{r1}}\sum_{v\in\mathcal{V}_{r2}} (\xi_u\xi_v e_u e_v^T + \xi_u\xi_v e_v e_u^T)$$
$$= \frac{1}{2} I_{r_0 n} + \frac{1}{2}(\xi\xi^T - \eta\eta^T) = \xi\xi^T + \frac{1}{2}\Gamma.$$

Recall that $e_i^s = e_i^{(s_0 n)}$. Similarly to step 1, we have that

$$\mathbb{E}\{Q(t)^T\xi\xi^T R(t)\} = \frac{c_s\lambda_1}{l^{(s)}}\xi\tilde{\xi}^T - \frac{\lambda_1}{2r_0 n l^{(s)}}\tilde{M},$$

where we denote

$$\tilde{\xi} := \frac{1}{\sqrt{s_0 n}} \sum_{1\le v\le s_0 n}\Big(\sum_{u\in\mathcal{V}_{r1}} l_{uv}^{(s)} - \sum_{u\in\mathcal{V}_{r2}} l_{uv}^{(s)}\Big) e_v^s.$$

In addition, $\mathbb{E}\{R(t)^T\xi\xi^T R(t)\} = \lambda_1\tilde{D}^{(s)}/(2r_0 n l^{(s)})$. Therefore,

$$\text{(A.5)} = (1-2\lambda_2)\|X^\xi(t)\|^2 + \frac{\lambda_1}{2r_0 n}\|X^\eta(t)\|^2 + \frac{4\lambda_2-3\lambda_1}{2r_0 n}\|X^\xi(t)\|^2 + \frac{2\lambda_2-\lambda_1}{2r_0 n}\|X^\Gamma(t)\|^2 + \frac{2c_s\lambda_1}{l^{(s)}}X(t)^T\xi\tilde{\xi}^T z^s - \frac{\lambda_1}{r_0 n l^{(s)}}X(t)^T\tilde{M}z^s + \frac{\lambda_1}{2r_0 n l^{(s)}}(z^s)^T\tilde{D}^{(s)}z^s.$$

Hence when $n \ge 4/r_0$, the following bounds hold

$$\mathbb{E}\{\|X^\xi(t+1)\|^2\} \le \Big(1-\frac{3}{2}\lambda_2\Big)\mathbb{E}\{\|X^\xi(t)\|^2\} + \frac{\lambda_1}{2r_0 n}\mathbb{E}\{\|X^\eta(t)\|^2\} + \frac{\lambda_2}{r_0 n}\mathbb{E}\{\|X^\Gamma(t)\|^2\} + \frac{2c_s\lambda_1}{l^{(s)}}\mathbb{E}\{X(t)\}^T\xi\tilde{\xi}^T z^s - \frac{\lambda_1}{r_0 n l^{(s)}}\mathbb{E}\{X(t)\}^T\tilde{M}z^s + \frac{\lambda_1\tilde{l}_+^{(s)}}{2r_0 n l^{(s)}}\|z^s\|^2, \tag{A.6}$$

$$\mathbb{E}\{\|X^\xi(t+1)\|^2\} \ge (1-2\lambda_2)\mathbb{E}\{\|X^\xi(t)\|^2\} + \frac{2c_s\lambda_1}{l^{(s)}}\mathbb{E}\{X(t)\}^T\xi\tilde{\xi}^T z^s - \frac{\lambda_1}{r_0 n l^{(s)}}\mathbb{E}\{X(t)\}^T\tilde{M}z^s. \tag{A.7}$$

Furthermore, by induction and from (A.6), Assumptions 2 and 3, and Lemma 5 at the end of this section,

$$\mathbb{E}\{\|X^\xi(t)\|^2\} \le (1-\lambda_2)^t\|X^\xi(0)\|^2 + \lambda_1\Big(\frac{1}{\lambda_2}\wedge t\Big) \Big[2c_s c_l(\|X^\xi(0)\| + |\zeta_2|)\|z^s\| + \frac{1}{2r_0 n}(c_x^2 r_0 n + \|X(0)\|^2 + 3\|\tilde{z}^s\|^2 + c_l\|z^s\|^2)\Big] + \lambda_2\Big(\frac{1}{\lambda_2}\wedge t\Big)c_x^2, \tag{A.8}$$

where $\zeta_2 := \xi^T\bar{R}z^s/\lambda_1 = \xi^T\tilde{M}z^s/l^{(s)}$.

Step 3 (Bounding $\mathbb{E}\{\|X^\Gamma(t)\|^2\}$). Note that

$$\mathbb{E}\{\|X(t+1)\|^2|X(t)\} = X(t)^T\mathbb{E}\{Q(t)^TQ(t)\}X(t) + 2X(t)^T\mathbb{E}\{Q(t)^TR(t)\}z^s + z^s\mathbb{E}\{R(t)^TR(t)\}z^s, \tag{A.9}$$

$$\mathbb{E}\{Q(t)^TQ(t)\} = \sum_{\{u,v\}\in\mathcal{E}_r}\Big[I - \frac{1}{2}(e_u-e_v)(e_u-e_v)^T\Big]\frac{a_{uv}}{\alpha} + \sum_{\{u,v\}\in\mathcal{E}_s}\Big(I - \frac{3}{4}e_u e_u^T\Big)\frac{a_{uv}}{\alpha}$$
$$= \bar{Q} - \frac{\lambda_1}{2}I_{r_0 n}$$
$$= \Big(1-\frac{3}{2}\lambda_1\Big)\eta\eta^T + \Big(1-\lambda_2-\frac{1}{2}\lambda_1\Big)\xi\xi^T + \Big(1-\lambda_3-\frac{1}{2}\lambda_1\Big)\Gamma,$$

$$\mathbb{E}\{Q(t)^TR(t)\} = \sum_{1\le v\le s_0 n}\sum_{u\in\mathcal{V}_r}\Big(I-\frac{1}{2}e_u e_u^T\Big)\Big(\frac{1}{2}e_u(e_v^s)^T\Big)\frac{l_{uv}^{(s)}}{\alpha} = \frac{\lambda_1\tilde{M}}{2l^{(s)}},$$

$$\mathbb{E}\{R(t)^TR(t)\} = \frac{1}{4\alpha}\sum_{1\le v\le s_0 n}\sum_{u\in\mathcal{V}_r} e_v^s(e_v^s)^T l_{uv}^{(s)} = \frac{\lambda_1\tilde{D}^{(s)}}{2l^{(s)}}.$$

Thus,

$$\text{(A.9)} = \Big(1-\frac{3}{2}\lambda_1\Big)\|X^\eta(t)\|^2 + \Big(1-\lambda_2-\frac{1}{2}\lambda_1\Big)\|X^\xi(t)\|^2 + \Big(1-\lambda_3-\frac{1}{2}\lambda_1\Big)\|X^\Gamma(t)\|^2 + \frac{\lambda_1}{l^{(s)}}X(t)^T\tilde{M}z^s + \frac{\lambda_1}{2l^{(s)}}(z^s)^T\tilde{D}^{(s)}z^s. \tag{A.10}$$

It follows from the preceding equation, Lemma 1(ii), (A.3) and (A.7) that

$$\mathbb{E}\{\|X^\Gamma(t+1)\|^2\} \le \lambda_1\mathbb{E}\{\|X^\eta(t)\|^2\} + \lambda_2\mathbb{E}\{\|X^\xi(t)\|^2\} + (1-\lambda_3)\mathbb{E}\{\|X^\Gamma(t)\|^2\} + \Big(1+\frac{2}{r_0 n}\Big)\frac{\lambda_1}{l^{(s)}}\mathbb{E}\{X(t)\}^T\tilde{M}z^s + \frac{\lambda_1\tilde{l}_+^{(s)}}{2l^{(s)}}\|z^s\|^2 - \frac{2c_s\lambda_1}{l^{(s)}}(\mathbb{E}\{X(t)\}^T\eta\tilde{\eta}^T z^s + \mathbb{E}\{X(t)\}^T\xi\tilde{\xi}^T z^s).$$

From Lemma 5, Assumptions 2 and 3, and (A.8), we know by induction that

$$\mathbb{E}\{\|X^\Gamma(t)\|^2\} \le \lambda_2\mathbb{E}\{\|X^\xi(t-1)\|^2\} + (1-\lambda_3)\mathbb{E}\{\|X^\Gamma(t-1)\|^2\} + \lambda_1 c_x^2 r_0 n + \Big(1+\frac{2}{r_0 n}\Big)\frac{\lambda_1}{2}(\|X(0)\|^2 + 3\|\tilde{z}^s\|^2) + \frac{c_l\lambda_1}{2}\|z^s\|^2 + 2c_s c_l\lambda_1\|z^s\|(\|X^\eta(0)\| + \|X^\xi(0)\| + |\zeta_1| + |\zeta_2|)$$
$$\le (1-\lambda_3)^t\|X^\Gamma(0)\|^2 + \lambda_1\Big(\frac{1}{\lambda_3}\wedge t\Big)\Big[\Big(1+\frac{1}{2r_0 n}\Big)c_x^2 r_0 n + \frac{1}{2}\Big(1+\frac{3}{r_0 n}\Big)(\|X(0)\|^2 + 3\|\tilde{z}^s\|^2) + \Big(1+\frac{1}{r_0 n}\Big)\frac{c_l}{2}\|z^s\|^2 + 2c_s c_l\|z^s\|(\|X^\eta(0)\| + 2\|X^\xi(0)\| + |\zeta_1| + 2|\zeta_2|)\Big] + \lambda_2\Big(\frac{1}{\lambda_3}\wedge t\Big)[(1-\lambda_2)^t\|X^\xi(0)\|^2 + c_x^2]. \tag{A.11}$$

Step 4 (Putting everything together). Lemma 1 yields that $\mathbb{E}\{\|X(t)\|^2\} = \mathbb{E}\{\|X^\eta(t)\|^2\} + \mathbb{E}\{\|X^\xi(t)\|^2\} + \mathbb{E}\{\|X^\Gamma(t)\|^2\}$, so (5) follows from summarizing (A.4), (A.8), (A.11), and the fact $1/\lambda_1 \ge 1/\lambda_k$, $k = 2, 3$.

Proof of (6). The derivation of the upper bound (6) is similar to that of (5). Decompose $X(t) = X^\eta(t) + X^\perp(t)$. In step 1, we have obtained upper and lower bounds for $\mathbb{E}\{\|X^\eta(t)\|^2\}$, so it suffices to obtain an upper bound for $\mathbb{E}\{\|X^\perp(t)\|^2\}$. From (A.3) and (A.10),

$$\mathbb{E}\{\|X^\perp(t+1)\|^2\} \le \Big(1-\frac{3}{2}\lambda_1\Big)\mathbb{E}\{\|X^\eta(t)\|^2\} + \Big(1-\lambda_2\wedge\lambda_3-\frac{1}{2}\lambda_1\Big)\mathbb{E}\{\|X^\perp(t)\|^2\} + \frac{\lambda_1}{l^{(s)}}\mathbb{E}\{X(t)\}^T\tilde{M}z^s + \frac{\lambda_1}{2l^{(s)}}(z^s)^T\tilde{D}^{(s)}z^s - \mathbb{E}\{\|X^\eta(t+1)\|^2\}$$
$$\le (1-\lambda_2\wedge\lambda_3)\mathbb{E}\{\|X^\perp(t)\|^2\} + \lambda_1 c_x^2 r_0 n\Big(3 + \frac{c_l}{2} + 4c_s c_l + \frac{2}{r_0 n}\Big)$$
$$\le (1-\lambda_2\wedge\lambda_3)^t\|X^\perp(0)\|^2 + \lambda_1\Big(\frac{1}{\lambda_2\wedge\lambda_3}\wedge t\Big)c_x^2 r_0 n \Big(3 + \frac{c_l}{2} + 4c_s c_l + \frac{2}{r_0 n}\Big). \tag{A.12}$$

Therefore, the bound (6) follows from combining the preceding inequality and (A.4). □

Lemma 5. *Under the conditions of Lemma 4, the following bounds hold,*

$$|\mathbb{E}\{X(t)\}^T\eta\tilde{\eta}^Tz^s| \le \tilde{l}_+^{(s)}(\|X^\eta(0)\| + |\zeta_1|)\|z^s\|,$$
$$|\mathbb{E}\{X(t)\}^T\xi\tilde{\xi}^Tz^s| \le \tilde{l}_+^{(s)}(\|X^\xi(0)\| + |\zeta_2|)\|z^s\|,$$
$$\left|\frac{\mathbb{E}\{X(t)\}^T\tilde{M}z^s}{l^{(s)}}\right| \le \frac{1}{2}(\|X(0)\|^2 + 3\|\tilde{z}^s\|^2).$$

Proof. It follows from Lemma 3 that, for all $t \in \mathbb{N}$,

$$\begin{aligned}
&|\mathbb{E}\{X(t)\}^T\eta\tilde{\eta}^Tz^s| \\
&= \left|\left\{(1-\lambda_1)^t\eta^TX(0) + \frac{1}{\lambda_1}[1-(1-\lambda_1)^t]\eta^T\bar{R}z^s\right\}\tilde{\eta}^Tz^s\right| \\
&\le (1-\lambda_1)^t|\eta^TX(0)\tilde{\eta}^Tz^s| + [1-(1-\lambda_1)^t]\left|\eta^T\frac{\bar{R}}{\lambda_1}z^s\tilde{\eta}^Tz^s\right| \\
&\le |\eta^TX(0)\tilde{\eta}^Tz^s| + \left|\eta^T\frac{\bar{R}}{\lambda_1}z^s\tilde{\eta}^Tz^s\right| \\
&\le \tilde{l}_+^{(s)}(\|X^\eta(0)\| + |\zeta_1|)\|z^s\|, \\
&|\mathbb{E}\{X(t)\}^T\xi\tilde{\xi}^Tz^s| \\
&= \left|\left\{(1-\lambda_2)^t\xi^TX(0) + \frac{1}{\lambda_2}[1-(1-\lambda_2)^t]\xi^T\bar{R}z^s\right\}\tilde{\xi}^Tz^s\right| \\
&\le (\|X^\xi(0)\| + |\zeta_2|)\|\tilde{\xi}\|\|z^s\| \le \tilde{l}_+^{(s)}(\|X^\xi(0)\| + |\zeta_2|)\|z^s\|.
\end{aligned}$$

Again from Lemma 3, it holds that

$$\begin{aligned}
&\left|\frac{1}{l^{(s)}}(\mathbb{E}\{X(t)\})^T\tilde{M}z^s\right| \\
&= \Big|(1-\lambda_1)^t\eta^TX(0)\eta^T\frac{\tilde{M}}{l^{(s)}}z^s + [1-(1-\lambda_1)^t]\eta^T\frac{\bar{R}}{\lambda_1}z^s\eta^T\frac{\tilde{M}}{l^{(s)}}z^s \\
&+ (1-\lambda_2)^t\xi^TX(0)\xi^T\frac{\tilde{M}}{l^{(s)}}z^s + \frac{\lambda_1}{\lambda_2}[1-(1-\lambda_2)^t]\xi^T\frac{\bar{R}}{\lambda_1}z^s\xi^T\frac{\tilde{M}}{l^{(s)}}z^s \\
&+ (1-\lambda_3)^t\sum_{i=3}^{r_0n}(w^{(i)})^TX(0)(w^{(i)})^T\frac{\tilde{M}}{l^{(s)}}z^s \\
&+ \frac{\lambda_1}{\lambda_3}[1-(1-\lambda_3)^t]\sum_{i=3}^{r_0n}(w^{(i)})^T\frac{\bar{R}}{\lambda_1}z^s(w^{(i)})^T\frac{\tilde{M}}{l^{(s)}}z^s\Big| \\
&\le \left|\eta^TX(0)\eta^T\frac{\tilde{M}}{l^{(s)}}z^s\right| + \left|\eta^T\frac{\bar{R}}{\lambda_1}z^s\eta^T\frac{\tilde{M}}{l^{(s)}}z^s\right| + \left|\xi^TX(0)\xi^T\frac{\tilde{M}}{l^{(s)}}z^s\right| \\
&+ \left|\xi^T\frac{\bar{R}}{\lambda_1}z^s\xi^T\frac{\tilde{M}}{l^{(s)}}z^s\right| + \left|\sum_{i=3}^{r_0n}(w^{(i)})^TX(0)(w^{(i)})^T\frac{\tilde{M}}{l^{(s)}}z^s\right| \\
&+ \left|\sum_{i=3}^{r_0n}(w^{(i)})^T\frac{\bar{R}}{\lambda_1}z^s(w^{(i)})^T\frac{\tilde{M}}{l^{(s)}}z^s\right| \\
&\le (\|X^\eta(0)\|\ |\zeta_1| + |\zeta_1|^2 + \|X^\xi(0)\|\ |\zeta_2| + |\zeta_2|^2 \\
&+ \|X^\Gamma(0)\|\ |\zeta_3| + |\zeta_3|^2) \qquad \text{(A.13)} \\
&\le \frac{1}{2}[\|X^\eta(0)\|^2 + \|X^\xi(0)\|^2 + \|X^\Gamma(0)\|^2 + 3(|\zeta_1|^2 + |\zeta_2|^2 + |\zeta_3|^2)] \\
&\qquad \text{(from } 2ab \le a^2 + b^2, \forall a, b \in \mathbb{R}) \\
&= \frac{1}{2}(\|X(0)\|^2 + 3\|\tilde{z}^s\|^2).
\end{aligned}$$

Here, the last equation follows from Lemma 1(ii), and (A.13) is obtained from the following fact,

$$\begin{aligned}
&\left|\sum_{i=3}^{r_0n}(w^{(i)})^TX(0)(w^{(i)})^T\frac{\tilde{M}}{l^{(s)}}z^s\right| \\
&\le \left(\sum_{i=3}^{r_0n}((w^{(i)})^TX(0))^2\right)^{\frac{1}{2}}\left(\sum_{i=3}^{r_0n}\left((w^{(i)})^T\frac{\tilde{M}}{l^{(s)}}z^s\right)^2\right)^{\frac{1}{2}} \\
&= \left\|\sum_{i=3}^{r_0n}w^{(i)}(w^{(i)})^TX(0)\right\|\ \left\|\sum_{i=3}^{r_0n}w^{(i)}(w^{(i)})^T\frac{\tilde{M}}{l^{(s)}}z^s\right\| \\
&=: \|X^\Gamma(0)\|\ |\zeta_3|,
\end{aligned}$$

where the first inequality follows from the Cauchy–Schwarz inequality, and the first equation is from the orthogonality of $w^{(i)}$, $3 \le i \le r_0n$. □

Appendix B. Proof of Theorem 2

Similar to Theorem 1, to prove (i) it suffices to bound the probability $\mathbb{P}\{|\mathcal{S}(X(t), X^\eta(0) + X^\xi(0), \varepsilon)| \ge \delta r_0 n\}$. From Lemma 3 and the Bernoulli inequality we have that

$$\begin{aligned}
&\mathbb{E}\{\|X(t) - (X^\eta(0) + X^\xi(0))\|^2\} \\
&= \mathbb{E}\{\|X(t)\|^2\} + \|X^\eta(0) + X^\xi(0)\|^2 - 2\mathbb{E}\{X(t)\}^T(X^\eta(0) + X^\xi(0)) \\
&= \mathbb{E}\{\|X(t)\|^2\} + \|X^\eta(0)\|^2 + \|X^\xi(0)\|^2 - 2(1-\lambda_1)^t\|X^\eta(0)\|^2 \\
&- [1-(1-\lambda_1)^t]\zeta_1\eta^TX(0) - 2(1-\lambda_2)^t\|X^\xi(0)\|^2 \\
&- \frac{\lambda_1}{\lambda_2}[1-(1-\lambda_2)^t]\zeta_2\xi^TX(0) \\
&\le \mathbb{E}\{\|X(t)\|^2\} + [1-2(1-\lambda_1)^t]\|X^\eta(0)\|^2 \\
&+ [1-2(1-\lambda_2)^t]\|X^\xi(0)\|^2 + 2\lambda_1t|\zeta_1|\|X^\eta(0)\| + 2\lambda_1t|\zeta_2|\|X^\xi(0)\| \\
&\le \mathbb{E}\{\|X(t)\|^2\} + [1-2(1-\lambda_1)^t]\|X^\eta(0)\|^2 \\
&+ [1-2(1-\lambda_2)^t]\|X^\xi(0)\|^2 + 2\lambda_1tc_x^2r_0n \\
&\le (1-\lambda_3)^t\|X^\Gamma(0)\|^2 + \lambda_1tc_x^2r_0n\Big(6 + \frac{c_l}{2} + 16c_sc_l + \frac{3c_l+23}{2r_0n}\Big) \\
&+ \Big(\frac{\lambda_2}{\lambda_3} + \lambda_2t\Big)(\|X^\xi(0)\|^2 + c_x^2),
\end{aligned}$$

where the last inequality follows from (5) with $1/\lambda_3 \le t \le 1/\lambda_2$. Hence from (9),

$$\begin{aligned}
&\mathbb{P}\{|\mathcal{S}(X(t), X^\eta(0) + X^\xi(0), \varepsilon)| \ge \delta r_0 n\} \\
&\le \frac{1}{\varepsilon^2\delta}\Big[(1-\lambda_3)^t + (3 + C_{11})\lambda_1t + C_{12}\Big(\frac{\lambda_2}{\lambda_3} + \lambda_2t\Big)\Big],
\end{aligned}$$

and the conclusion of (i) follows. The second part of the theorem can be derived from similar calculations.

Appendix C. Proof of Theorem 3

Denote the maximum-absolute-column-sum, spectral, and maximum-absolute-row-sum norm by $\|\cdot\|_1$, $\|\cdot\|$, and $\|\cdot\|_\infty$. Let $\{[Q'(t)\ U'(t)], t \in \mathbb{N}\}$ be an i.i.d. sequence having the same distribution as and independent of the sequence $\{[Q(t)\ U(t)], t \in \mathbb{N}\}$ with $U(t) := R(t)z^s$. Denote $\Phi_{Q'}(s,t) = Q'(t)\cdots Q'(s)$, $\overleftarrow{\Phi}_{Q'}(s,t) = Q'(s)\cdots Q'(t)$, $\Phi_{Q'}(t+1,t) = \overleftarrow{\Phi}_{Q'}(t+1,t) = I$, $t \ge s \ge 0$, and

$$\tilde{X}(t) = \Phi_{Q'}(0,t)X(0) + \sum_{i=0}^{t}\Phi_{Q'}(t+1-i,t)U'(t-i),$$

$$\tilde{X}^*(t) = \sum_{i=0}^{t}\Phi_{Q'}(t+1-i,t)U'(t-i) + \Phi_{Q'}(0,t)\sum_{i=t+1}^{\infty}\overleftarrow{\Phi}_{Q'}(t+1,i-1)U'(i).$$

So $\tilde{X}(t)$ and $\tilde{X}^*(t)$ have the same distribution as $X(t)$ and π, respectively (Acemoğlu et al., 2013; Xing et al., 2023). Hence, the following result yields the conclusion.

$$
\begin{aligned}
& d_W(X(t), \pi) \leq \mathbb{E}\{\|\tilde{X}(t) - \tilde{X}^*(t)\|\} \\
&= \mathbb{E}\Big\{ \Big\| \Phi_{Q'}(0,t)X(0) - \Phi_{Q'}(0,t) \sum_{i=t+1}^{\infty} \overleftarrow{\Phi}_{Q'}(t+1, i-1)U'(i) \Big\| \Big\} \\
&\leq c_x \sqrt{r_0 n}\,\mathbb{E}\{\|\Phi_{Q'}(0,t)\|\} \\
&\quad + \frac{1}{2} c_x \sum_{i=t+1}^{\infty} \mathbb{E}\{\|\Phi_{Q'}(0,t)\overleftarrow{\Phi}_{Q'}(t+1, i-1)\|\} \\
&\leq \frac{1}{2} c_x r_0 n \Big(2\mathbb{E}\{\|\Phi_{Q'}(0,t)\|_1\} \\
&\quad + \sum_{i=t+1}^{\infty} \mathbb{E}\{\|\Phi_{Q'}(0,t)\overleftarrow{\Phi}_{Q'}(t+1, i-1)\|_1\} \Big) \\
&\leq c_x (r_0 n)^{\frac{5}{2}} \Big(1 + \frac{1}{2\lambda_1}\Big)(1-\lambda_1)^{t+1},
\end{aligned}
$$

where the second inequality follows from Assumption 2, and the last inequality is obtained from

$$
\begin{aligned}
& \mathbb{E}\{\|\overleftarrow{\Phi}_Q(0,t)\|_1\} \leq \sum_{1 \leq j \leq r_0 n} \sum_{1 \leq i \leq r_0 n} \mathbb{E}\{|[\overleftarrow{\Phi}_Q(0,t)]_{ij}|\} \\
&\leq r_0 n \|\mathbb{E}\{\overleftarrow{\Phi}_Q(0,t)\}\|_\infty = r_0 n \|\bar{Q}^{t+1}\|_\infty \leq (r_0 n)^{\frac{3}{2}} (1-\lambda_1)^{t+1}.
\end{aligned}
$$

References

Abbe, E. (2017). Community detection and stochastic block models: Recent developments. *Journal of Machine Learning Research*, *18*(1), 6446–6531.

Abelson, R. P. (1964). Mathematical models of the distribution of attitudes under controversy. *Contributions to Mathematical Psychology*.

Acemoğlu, D., Como, G., Fagnani, F., & Ozdaglar, A. (2013). Opinion fluctuations and disagreement in social networks. *Mathematics of Operations Research*, *38*(1), 1–27.

Banisch, S., & Araújo, T. (2010). On the empirical relevance of the transient in opinion models. *Physics Letters. A*, *374*(31–32), 3197–3200.

Banisch, S., Lima, R., & Araújo, T. (2012). Agent based models and opinion dynamics as Markov chains. *Social Networks*, *34*(4), 549–561.

Barbillon, P., Thomas, M., Goldringer, I., Hospital, F., & Robin, S. (2015). Network impact on persistence in a finite population dynamic diffusion model: Application to an emergent seed exchange network. *Journal of Theoretical Biology*, *365*, 365–376.

Baumann, F., Sokolov, I. M., & Tyloo, M. (2020). A Laplacian approach to stubborn agents and their role in opinion formation on influence networks. *Physica A. Statistical Mechanics and its Applications*, *557*, Article 124869.

Boyd, S., Ghosh, A., Prabhakar, B., & Shah, D. (2006). Randomized gossip algorithms. *IEEE Transaction on Information Theory*, *52*(6), 2508–2530.

Castellano, C., Fortunato, S., & Loreto, V. (2009). Statistical physics of social dynamics. *Reviews of Modern Physics*, *81*(2), 591.

Cheng, X., Kawano, Y., & Scherpen, J. M. (2018). Model reduction of multiagent systems using dissimilarity-based clustering. *IEEE Transactions on Automatic Control*, *64*(4), 1663–1670.

Chowell, G., Sattenspiel, L., Bansal, S., & Viboud, C. (2016). Mathematical models to characterize early epidemic growth: A review. *Physics of Life Reviews*, *18*, 66–97.

Chung, F., & Radcliffe, M. (2011). On the spectra of general random graphs. *The Electronic Journal of Combinatorics*, P215.

Como, G., & Fagnani, F. (2016). From local averaging to emergent global behaviors: The fundamental role of network interconnections. *Systems & Control Letters*, *95*, 70–76.

Conover, M., Ratkiewicz, J., Francisco, M., Gonçalves, B., Menczer, F., & Flammini, A. (2011). Political polarization on Twitter. *Vol. 5*, In *Proceedings of the International AAAI Conference on Web and Social Media* (pp. 89–96).

Cota, W., Ferreira, S. C., Pastor-Satorras, R., & Starnini, M. (2019). Quantifying echo chamber effects in information spreading over political communication networks. *EPJ Data Science*, *8*(1), 1–13.

De, A., Bhattacharya, S., Bhattacharya, P., Ganguly, N., & Chakrabarti, S. (2019). Learning linear influence models in social networks from transient opinion dynamics. *ACM Transactions on the Web*, *13*(3), 1–33.

Deffuant, G., Neau, D., Amblard, F., & Weisbuch, G. (2000). Mixing beliefs among interacting agents. *Advances in Complex Systems*, *3*(01n04), 87–98.

DeGroot, M. H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, *69*(345), 118–121.

Dietrich, F., Martin, S., & Jungers, M. (2016). Transient cluster formation in generalized Hegselmann-Krause opinion dynamics. In *European Control Conference* (pp. 531–536).

Fagnani, F., & Zampieri, S. (2008). Randomized consensus algorithms over large scale networks. *IEEE Journal on Selected Areas in Communications*, *26*(4), 634–649.

Fennell, S. C., Burke, K., Quayle, M., & Gleeson, J. P. (2021). Generalized mean-field approximation for the Deffuant opinion dynamics model on networks. *Physical Review E*, *103*(1), Article 012314.

Festinger, L. (1949). The analysis of sociograms using matrix algebra. *Human Relations*, *2*(2), 153–158.

Flache, A., Mäs, M., Feliciani, T., Chattoe-Brown, E., Deffuant, G., Huet, S., & Lorenz, J. (2017). Models of social influence: Towards the next frontiers. *Journal of Artificial Societies and Social Simulation*, *20*(4).

Fortunato, S., & Hric, D. (2016). Community detection in networks: A user guide. *Physics Reports*, *659*, 1–44.

Friedkin, N. E., & Johnsen, E. C. (1990). Social influence and opinions. *Journal of Mathematical Sociology*, *15*(3–4), 193–206.

Friedkin, N. E., Proskurnikov, A. V., & Bullo, F. (2021). Group dynamics on multidimensional object threat appraisals. *Social Networks*, *65*, 157–167.

Gargiulo, F., & Huet, S. (2010). Opinion dynamics in a group-based society. *Europhysics Letters*, *91*(5), 58004.

Hegselmann, R., & Krause, U. (2002). Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, *5*(3).

Hill, S. J., Lo, J., Vavreck, L., & Zaller, J. (2013). How quickly we forget: The duration of persuasion effects from mass communication. *Political Communication*, *30*(4), 521–547.

Holme, P., & Newman, M. E. (2006). Nonequilibrium phase transition in the coevolution of networks and opinions. *Physical Review E*, *74*(5), Article 056108.

Kozitsin, I. V. (2023). Opinion dynamics of online social network users: A micro-level analysis. *The Journal of Mathematical Sociology*, *47*(1), 1–41.

Lorenz, J. (2006). Consensus strikes back in the Hegselmann-Krause model of continuous opinion dynamics under bounded confidence. *Journal of Artificial Societies and Social Simulation*, *9*(1), 8.

Newman, M. E., & Girvan, M. (2004). Finding and evaluating community structure in networks. *Physical Review E*, *69*(2), Article 026113.

Noorazar, H., Vixie, K. R., Talebanpour, A., & Hu, Y. (2020). From classical to modern opinion dynamics. *International Journal of Modern Physics C*, *31*(07), Article 2050101.

Proskurnikov, A. V., & Tempo, R. (2018). A tutorial on modeling and analysis of dynamic social networks. Part II. *Annual Reviews in Control*, *45*, 166–190.

Schaub, M. T., Segarra, S., & Tsitsiklis, J. N. (2020). Blind identification of stochastic block models from dynamical observations. *SIAM Journal on Mathematics of Data Science*, *2*(2), 335–367.

Shi, G., Altafini, C., & Baras, J. S. (2019). Dynamics over signed networks. *SIAM Review*, *61*(2), 229–257.

Shi, G., Proutiere, A., Johansson, M., Baras, J. S., & Johansson, K. H. (2016). The evolution of beliefs over signed social networks. *Operations Research*, *64*(3), 585–604.

Shree, S. S., Avhishek, C., & Krishna, J. (2022). Finite time bounds for stochastic bounded confidence dynamics. arXiv preprint arXiv:2212.13387.

Si, X., Liu, Y., & Zhang, Z. (2009). Opinion dynamics in populations with implicit community structure. *International Journal of Modern Physics C*, *20*(12), 2013–2026.

Xing, Y., He, X., Fang, H., & Johansson, K. H. (2023). Community structure recovery and interaction probability estimation for gossip opinion dynamics. *Automatica*, *154*, Article 111105.

Xing, Y., & Johansson, K. H. (2023). What is the expected transient behavior of opinion evolution for two communities? arXiv preprint arXiv:2304.12495.

Xiong, F., Liu, Y., & Cheng, J. (2017). Modeling and predicting opinion formation with trust propagation in online social networks. *Communications in Nonlinear Science and Numerical Simulation*, *44*, 513–524.

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