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How can an influencer maximize her social power?

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***Abstract*— This article considers Friedkin-Johnsen (FJ) model with an external influencer. The influencer is totally stubborn (i.e., her own opinion never changes), and can participate in the opinion evolution by adding links to a fixed number of agents in the FJ model. The problem is to investigate how the influencer selects agents to maximize her social power, which represents the influence of her initial opinion on other agents' final opinions. This problem is shown to be equivalent to maximize the absorbing probability to the influencer for a Markov chain. The solution is analytically characterized for the cases of small and large stubbornness, with a single agent to be selected. Moreover, the social power of the influencer is proved to be monotone and submodular with the selected agent set, and a greedy algorithm is then proposed to generate an approximated solution. Random walks are used to speed up the algorithm for large network size. The effectiveness of the greedy algorithm is further shown by a numerical example.**

I. INTRODUCTION

People's opinions are consistently influenced by their daily contacts with others, which are usually described by a social network. Opinion dynamics is to study how opinions evolve over social networks, and has attracted many research interests in the past decades [9], [14], [10], [2], [28], [33]. Various opinion dynamics models are proposed in literature, and a large part of them use a weight-averaging opinion update rule, which originates from the famous DeGroot model [9]. Particularly, in the Friedkin-Johnsen (FJ) model, the opinion of each agent is updated as the weighted-average of its neighbors' opinions and the initial opinion of its own, in which the weight assigned to initial opinion represents the agent's stubbornness. The FJ model can not only be used to describe opinion exchanges among small-group individuals [10], but also be applied to study complex negotiation process such as the UN climate change conferences [6], [5].

On the other hand, as opinions evolve over social networks, the influence or leadership of individuals also change. For example, if one's opinion is widely accepted by others, she will potentially be able to influence the behavior of other people, and obtain a high leadership in the group. This provides a natural incentive for people to increase their influence on others' opinions. Real-world examples can be that a political leader campaigns for more support of her proposition, a government tries to convince the public of the necessity of taking non-pharmaceutical precautions, etc.

Studies on social influence maximization has been an active research area since the beginning of this century [18]. There are mainly two types of work in literature, differing in the opinion models that are used, and correspondingly, the meaning of influence. The first type of work is based on information diffusion models such as the independent cascade (IC) model and the linear threshold (LT) model [20], [4], [7], [19]. In its basic formulation, an external influencer initially selects some nodes as "seeds" to inject information, then the information carried by the seeds eventually infects other nodes via social interactions, and the influencer's influence is measured by the number of final infected nodes. The second type of work is based on DeGroot-like models [15], [30], [31], [12], [29], [32], [3]. In such problems, the influencer is normally a totally stubborn agent that can select some followers to flip their initial opinions or add links to them. The influencer can consider various objectives, based on the group's final opinion values, for instance, altering the average final opinion, reducing opinion polarization [8], [25].

In this paper, we consider the scenario that an external influencer can select a group of agents to add links and influence the opinion evolution in an FJ model. The influencer is assumed to be totally stubborn. Instead of considering specific opinion values as in existing literature, the influencer wants to maximize its social power, which represents her ability to alter the average final opinions of the whole group, and can be regarded as a special type of homonic influcence centrality [26]. In this formulation, the influencer cares more about her longer-term leadership in possibly multiple FJ dynamics, instead of a one-shot alteration of group opinions.

The main results of this paper are presented in three parts. First, the social power maximization problem is shown to be equivalent to maximize the absorbing probability to the influencer in the Markov chain over an augmented graph (Lemma 2). Secondly, for the Markov chain, if the agent stubbornness is homogenous, the problem is equivalent to minimize the hitting time to the influencer (Lemma 3). Moreover, if only one agent is allowed to be selected, when the stubbornness is small, the agent should have the smallest weighted sum of the average expected hitting time from the other agents and returning time from herself over the original graph (Lemma 4); when the stubbornness is large, the agent should be the one with smallest weighted degree (Lemma 5). Lastly, the social power of the influencer is proved to be monotone and submodular with the set of selected agents (Propositions 1 and 2). As a consequence, a greedy algorithm

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is proposed to provide an approximated solution to the social power maximization problem (Algorithm 1 and Theorem 1). Furthermore, for large network size, random walks are used to approximate the matrix inverse that are required to be computed in the greedy algorithm (Algorithm 2), and a high probability bound of the algorithm is provided in Theorem 2.

The paper is organized as follows: Section II gives some preliminary knowledge and formulates the problem, and the main technical results and algorithms are reported in Section III. For the lack of space, all proofs are omitted and can be seen in the extended version [27].

Notation. All vectors are real column vectors and are denoted with bold lowercase letters $\mathbf{x}, \mathbf{y}, \dots$ The i-th entry of a vector $\mathbf{x}$ is denoted by $[\mathbf{x}]_i$ or, if no confusion arises, x_i. Matrices are denoted with the capital letters such as $A, B, \dots,$ of entries A_{ij} or $[A]_{ij}$. The identity matrix is denoted by I_n, with dimension sometimes omitted, depending on the context. The n-order vector and matrix with all entries being 0 or 1 are denoted by $\mathbf{0}_n$ or $\mathbf{1}_n$, respectively with the dimensions omitted if there is no confusion. Let $\mathbf{e}_i$ be the vector with the ith entry as 1 and all the others as 0. Given a set $\mathcal{C}$, we use $|\mathcal{C}|$ to denote its cardinality. A square matrix A is called (row) stochastic if $A \geq \mathbf{0}$ and $\mathbf{1} = A\mathbf{1}$. For two number sequences $f(n)$ and $g(n)$, we write $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $|f(n)| < Cg(n)$ holds for all $n \in \mathbb{N}$. Given a random variable $X \in \mathbb{R}$, its expectation is denoted as $\mathbb{E}[X]$.

II. Preliminaries and problem formulation

A. FJ model

Consider a network with nodes (agents) indexed in $\mathcal{V} = \{1, 2, \dots, n\}$. It is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $\mathcal{E}$ is a set of ordered pairs of nodes and $(i, j) \in \mathcal{E}$ represents a link from node i to node j. The matrix W is a stochastic weight matrix, such that for any $i, j \in \mathcal{V}$, $(i, j) \in \mathcal{E}$ if and only if $w_{ji} > 0$. A (directed) *path* is a concatenation of directed links of $\mathcal{E}$. We say that node i is connected to node j if there is a directed path from i to j. The graph $\mathcal{G}$ is called *strongly connected* if any two nodes are connected to each other.

The FJ model is a DeGroot-like model for opinion dynamics in which some agents behave stubbornly, in the sense that they defend their positions while discussing with the other agents [11]. The more stubborn an agent is, the lower is the total weight placed by the agent on others' opinion. If n agents participate in a discussion, the FJ model is

$$\mathbf{x}(t+1) = (I - \Theta)W\mathbf{x}(t) + \Theta\mathbf{x}(0), \quad t = 0, 1, \dots \tag{1}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^\top \in \mathbb{R}^n$ is the opinion vector of the agents, W is a row-stochastic matrix, and $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_n)$, with $\theta_i \in [0, 1]$ representing the stubbornness of agent i. Stubbornness here means attachment of an agent to her own opinion, represented by the initial condition $\mathbf{y}(0)$ at the beginning of the discussion ($\theta_i = 0$ means agent i is not stubborn, $\theta_i = 1$ means a totally stubborn agent).

Lemma 1 *[22] Consider the FJ model* (1) *over the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. Assume $\theta_i \in [0, 1]$ for all $i = 1, \dots, n$. If for each $i \in \mathcal{V}$, either $\theta_i > 0$ or the agent i is linked by a path from some agent j with $\theta_j > 0$, then*

(a) *$(I - \Theta)W$ is Schur stable, i.e. $\rho((I - \Theta)W) < 1$,*
(b) *The matrix $P = (I - (I - \Theta)W)^{-1}\Theta$ is stochastic,*
(c) $\mathbf{x}(\infty) = \lim_{t \to +\infty} \mathbf{x}(t) = P\mathbf{y}(0)$.

For an FJ model with convergent opinions, social power is defined as follows [16], [24].

Definition 1 (Social power) *Let the assumptions in Lemma 1 hold. The social power of each agent i is $\mathrm{sp}_i := \frac{1}{n}\mathbf{1}^\top P\mathbf{e}_i$.*

From Lemma 1, the solution matrix P of an FJ model encodes the influence of each agent's initial opinion on the final opinion formation. Therefore, the social power of each agent i, defined in Definition 1 is indeed the overall influence of agent i's initial opinion on the group's final opinions.

B. Problem formulation

Consider a groups of agents $\mathcal{V} = \{1, 2, \dots, n\}$ over a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, with W as a stochastic matrix. The opinion evolution of the agents in $\mathcal{V}$ follows the FJ model (1). The following assumption is made throughout the paper.

Assumption 1 *The graph $\mathcal{G}$ is strongly connected.*

An external influencer, referred to as agent 0 in this article, can add links to a subset of agents $\mathcal{S} \subset \mathcal{V}$. The agent 0 is totally stubborn, i.e., $\theta_0 = 1$. The weights of the newly-added links are the same, that is, there exists $\omega \in [0, 1)$ such that $w_{i0} = \omega$ holds for all $i \in \mathcal{S}$. Correspondingly, for each $i \in \mathcal{S}$, the weight of each incoming link $(j, i) \in \mathcal{E}$ becomes $(1 - \omega)w_{ij}$. Let $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{W})$ be the augmented graph by adding agent 0 to the graph $\mathcal{G}$, with $\bar{V} = \mathcal{V} \cup \{0\}, \bar{\mathcal{E}} = \mathcal{E} \cup \{(0, i) : i \in \mathcal{S} \cup \{0\}\}$ and

$$\bar{W} = \begin{pmatrix} 1 & \mathbf{0} \\ \omega \sum_{j \in \mathcal{V}} \mathbf{e}_j & (I - \omega \sum_{j \in \mathcal{S}} \mathbf{e}_j \mathbf{e}_j^\top)W \end{pmatrix}. \tag{2}$$

The opinion dynamics with the influencer then becomes

$$\bar{\mathbf{x}}(t+1) = (I_{n+1} - \bar{\Theta})\bar{W}\bar{\mathbf{x}}(t) + \bar{\Theta}\bar{\mathbf{x}}(0), \tag{3}$$

with $\bar{\mathbf{x}}(t) = (x_0(t), x_1(t), \dots, x_n(t))^\top$ be the augmented opinion vector, and $\bar{\Theta} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \Theta \end{pmatrix}$. Under Assumption 1, from Lemma 1, the opinions converge,

$$\begin{aligned} \bar{\mathbf{x}}(\infty) &= \lim_{t \to \infty} \bar{\mathbf{x}}(t) = \bar{P}\bar{\mathbf{x}}(0) = (I - (I - \bar{\Theta})\bar{W})^{-1}\bar{\Theta}\bar{\mathbf{x}}(0) \\ &= \begin{pmatrix} 1 & 0 \\ \mathbf{p}_0 & P \end{pmatrix} \bar{\mathbf{x}}(0), \end{aligned}$$

with $P = (I - H(\mathcal{S}))^{-1}\Theta$ and

$$\begin{aligned} \mathbf{p}_0 &= (I - H(\mathcal{S}))^{-1}(I - \Theta)\omega \sum_{j \in \mathcal{S}} \mathbf{e}_j, \\ H(\mathcal{S}) &= \left(I - \omega \sum_{j \in \mathcal{S}} \mathbf{e}_j \mathbf{e}_j^\top \right)(I - \Theta)W. \end{aligned} \tag{4}$$

The social power of agent 0 is

$$\mathrm{sp}_0 = \frac{1}{n+1}\mathbf{1}^\top \mathbf{p}_0 + \frac{1}{n+1}. \tag{5}$$

The problem studied in this paper is stated as follows.

Problem. Given the weight matrix W, the stubbornness matrix Θ, and an integer budget $K \geq 1$, maximize the social power of agent 0 by adding links to no more than K agents. Equivalently, solve the following optimization problem, with the set $\mathcal{S}$ collecting the linked nodes,

$$\begin{aligned} &\underset{\mathcal{S}}{Maximize} \quad \mathrm{sp}_0(\mathcal{S}) \\ &s.t. \qquad |\mathcal{S}| \leq K \end{aligned} \tag{6}$$

Here we write sp_0 as $\mathrm{sp}_0(\mathcal{S})$ to show its dependence on $\mathcal{S}$.

Remark 1 The external influencer, agent 0, represents some opinion leader who insists her initial opinion, such as media outlets or companies promoting their products to consumers in the real world. Instead of directly altering the other agents' opinions, the influencer intentionally maximize the social power, which can be intuitively regarded as her long-term leadership in possibly multiple rounds of opinion formation processes that follow the FJ dynamics.

Remark 2 Mathematically, the problem (6) is to maximize the average opinion increase of the agents in $\mathcal{V}$ if agent 0 flips her opinion from 0 to 1 while $x_i(0) = 0, \forall i \in \mathcal{V}$, which is in fact harmonic influence centrality of agent 0 [26], [15]. Note that the problem of maximizing opinion sums by selecting followers and adding links to stubborn leaders are considered for the Degroot model in [15], [30], [31], [29]. Similar problems are also considered for the FJ model in [12], [29], [32], though with different assumptions: [12] studies the case of flipping the opinions of existing agents, [29] examines continuous dynamics with partially stubborn influencers, while [32] assumes an undirected network structure, differing from the setting in this paper.

III. Analysis and algorithm design

In this subsection, we first show the connection of social power with the absorbing probability of a Markov chain. Secondly, for some special cases of the agent stubbornness and link weight, we use the Markov chain to characterize the solution of the problem (6) analytically. Finally, for general setting of the parameters, we provide a greedy algorithm to compute an approximated solution.

A. Connection between social power and absorbing probability

For an FJ model, we can construct an (absorbed) Markov chain, such that the social power of each agent is equal to the associated absorbing probability of a random walk generated by the Markov chain, which is called augmented Markov chain of the FJ model in this paper.

Definition 2 (Augmented Markov chain) *Given an FJ model over the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, with the matrix of stubbornness as Θ, its augmented Markov chain, denoted as $\mathcal{Y}(\mathcal{G}, \Theta)$, is a Markov chain $\{Y_t : t \in \mathbb{Z}_{\geq 0}\}$ with*

- *State space $\mathcal{T} \cup \mathcal{A}$, with $\mathcal{T} = \mathcal{V}$ including the transient states and $\mathcal{A} = \{1', 2', \ldots, n'\}$ including the absorbing states,*
- *Transition matrix $T = \begin{bmatrix} (I-\Theta)W & \Theta \\ \mathbf{0} & I \end{bmatrix}$,*

where for $i \in \mathcal{V}$, the i-th column (row) of T corresponds to the transient state i, and the $(n+i)$-th state corresponds to the absorbing state i'.

In Definition 2, each state i' can be regarded as an auxiliary state of the state i, which corresponds to the initial opinion of agent i in the FJ model.

For the FJ model with external influencer 0, consider its augmented Markov chain $\mathcal{Y}(\bar{\mathcal{G}}, \bar{\Theta})$ (denoted as $\bar{\mathcal{Y}}$ in the following if no confusion arise) over the augmented graph $\bar{\mathcal{G}}$. For the simplicity of notation, we merge the states 0 and $0'$, and the transition matrix of $\bar{\mathcal{Y}}$ then becomes

$$T(\mathcal{S}) = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \omega(I-\Theta)\sum_{j\in\mathcal{S}} \mathbf{e}_j & H(\mathcal{S}) & \Theta \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}, \tag{7}$$

where the first column (row) corresponds to the absorbing state $0'$; for $i \in \mathcal{V}$, the $(i+1)$-th column (row) corresponds to the transient state i, and the $(n+i+1)$-th state corresponds to the absorbing state i'.

For the augmented Markov chain $\bar{\mathcal{Y}} = \{\bar{Y}_t : t \in \mathbb{Z}_{\geq 0}\}$, let the initial states uniformly distributed over the transient states and the absorbing state $0'$, i.e., $\mathbb{P}(\bar{Y}_0 = 0') = \mathbb{P}(\bar{Y}_0 = i) = \frac{1}{n+1}, \quad \forall i \in \mathcal{V}$. Let π_i be the absorbing probability of the state i', i.e., $\pi_i := \mathbb{P}(\bar{Y}_t = i' \text{ for some } t),\ i = 0, 1, \ldots, n$. Note that $(I - H(\mathcal{S}))^{-1}$ is the fundamental matrix of $\bar{\mathcal{Y}}$ [17], and correspondingly, $\pi_i = \frac{1}{n+1}\mathbf{1}^\top (I - H(\mathcal{S}))^{-1}\Theta \mathbf{e}_i = \frac{1}{n+1}\mathbf{1}^\top P \mathbf{e}_i = \mathrm{sp}_i, \quad \forall i \in \mathcal{V}$ and $\pi_0 = \mathrm{sp}_0$. Therefore, the following lemma can be directly obtained.

Lemma 2 *Consider the FJ model over the graph $\mathcal{G}$. Let Assumption 1 hold. The optimization problem* (6) *is equivalent to the following problem*

$$\underset{|\mathcal{S}|\leq K}{Maximize} \quad \pi_0(\mathcal{S}), \tag{8}$$

where $\pi_0(\mathcal{S})$ is the absorbing probability to $0'$ for the augmented Markov chain $\bar{\mathcal{Y}}$.

As mentioned in Remark 2, social power is a measurement of centrality. Lemma 2 provides an equivalent characterization of the centrality by using absorbing probabilities of a Markov chain. This also paves the way to the analysis in the next subsection. Note that in [12], a weighted sum of such absorbing probabilities is also shown to be equal to the sum of final opinions in the FJ model.

B. Influence of stubbornness and link weight ω

In this subsection we directly consider the problem (8), under the assumption that all the agents have homogenous stubbornness, i.e., for some $\theta \in (0, 1)$ it holds that $\theta_i =$

$\theta, \forall i \in \mathcal{V}$. For this case, the problem (8) is in fact to minimize hitting time of the augmented Markov chain to the absorbing states. Before introducing the following lemma, we define

$$f(\mathcal{S};\theta,\omega) := \frac{1}{n}\mathbf{1}^\top \left(I - (1-\theta)(I - \omega \sum_{j\in\mathcal{S}} \mathbf{e}_j\mathbf{e}_j^\top)W \right)^{-1} \mathbf{1}. \tag{9}$$

Lemma 3 *For the augmented Markov chain* $\bar{\mathcal{Y}} = \{\bar{Y}_t : t \in \mathbb{Z}_{\geq 0}\}$ *with* $\theta_i = \theta \in (0,1), \forall i \in \mathcal{V}$ *and* $\mathbb{P}(\bar{Y}_0 = i) = \frac{1}{n}, \forall i \in \mathcal{V}$*, define* $\bar{\tau} = \min_{t\geq 0}\{t : \bar{Y}_t \in \mathcal{A}\}$*. The optimization problem* (8) *is equivalent to*

$$\underset{|\mathcal{S}|=K}{Minimize} \quad \mathbb{E}_{\bar{\mathcal{Y}}}[\bar{\tau}], \tag{10}$$

which is also equivalent to $\underset{|\mathcal{S}|=K}{Minimize}\ f(\mathcal{S};\theta,\omega)$.

Remark 3 In the literature [1] and [13], similar problems of reducing hitting time for absorbing Markov chains are studied, under the scenario of reducing the so-called "structural bias" of networks. Compared to the literature, the problem (10) considers a more general network structure, in the sense that the transition probabilities from each node (or state) to its neighbors do not need to be equal. Moreover, the parameters θ and ω, not considered in [1] and [13], can affect the solution of the optimization problem (8) (or (10)), as discussed in what follows.

The matrix inverse in (9) makes it hard to disentangle the influence of the parameters θ and ω on the optimal selection of $\mathcal{S}$. Nevertheless, conclusions can be drawn for specific range of parameters. To simplify the discussion, the remaining part of this subsection assumes that $K = 1$, i.e., only one agent is selected by agent 0. Moreover, for any $i \in \mathcal{V}$, $f(\{i\};\theta,\omega)$ is simply written as $f(i;\theta,\omega)$.

We at first consider the case that θ is small enough, as specified by the following assumption.

Assumption 2 *Given* $\omega > 0$*,* θ *is small enough such that* $f(i;\theta,\omega) - f(i_*;\theta,\omega) > 0$ *holds for all* $i \neq i_*$*, where* $i_* = \arg\min_{j\in\mathcal{V}} f(j;0,\omega)$.

Assumption 2 requires that i_* is the only solution to the problem $Minimize_{i\in\mathcal{V}}\ f(i;0,\omega)$. As $f(i;\theta,\omega)$ is continuous with θ, there exists $\tilde{\theta} \in (0,1)$ such that for any $\theta < \tilde{\theta}$, Assumption 2 holds.

For the augmented Markov chain $\mathcal{Y}(\mathcal{G},\mathbf{0}) = \{Y_t : t \in \mathbb{Z}_{\geq 0}\}$ of the FJ model without agent 0, define

$$\tau_{ji} = \min\{t > 0 : Y_t = i | Y_0 = j\}$$

as the first time that the Markov chain hits $i \in \mathcal{V}$, given that the initial state is $j \in \mathcal{V}$.

Lemma 4 *Let Assumptions 1 and 2 hold. Consider the optimization problem* (8) *with* $K = 1$ *and* $\theta_i = \theta, \forall i \in \mathcal{V}$*. The problem is equivalent to*

$$\underset{i}{Minimize} \quad \frac{1}{n}\sum_{j\neq i} \mathbb{E}_{\mathcal{Y}(\mathcal{G},\mathbf{0})}[\tau_{ji}] + \frac{1-\omega}{\omega}\mathbb{E}_{\mathcal{Y}(\mathcal{G},\mathbf{0})}[\tau_{ii}] \tag{11}$$

Remark 4 The cost function of (11) is a weighted combination of two terms: the first term $\frac{1}{n}\sum_{j\neq i}\mathbb{E}_{\mathcal{Y}(\mathcal{G},\mathbf{0})}[\tau_{ji}]$ represents the average expected time for a random walk from the other agents to reach i, and the second term $\mathbb{E}_{\mathcal{Y}(\mathcal{G},\mathbf{0})}[\tau_{ii}]$ is the expected time from i to come back. Both terms only depend on the graph $\mathcal{G}$. For the problem (11) with a large ω (i.e., close to 1), it is more important to minimize the first term, which can be regarded as a measurement of the overall reachability to i in the graph; on the other hand, if ω is small (i.e., close to 0), the local structure of each agent in the graph affects more on the solution of (11).

Now consider the case that θ is close to 1. Denote $g_i := (1-\omega)W_{ii} + \sum_{j\neq i} W_{ji}$.

Assumption 3 *There exists only one agent* i^* *such that* $g_{i^*} = \max_{i\in\mathcal{V}} g_i$*. Moreover, for* $\delta_g := \min_{i\in\mathcal{V}}\{g_{i^*} - g_i : i \neq i^*\}$*, it holds* $\theta > \frac{n}{\delta_g + n}$.

Lemma 5 *Let Assumptions 1 and 3 hold. Consider the optimization problem* (8) *with* $K = 1$ *and* $\theta_i = \theta, \forall i \in \mathcal{V}$*. The solution to* (8) *is* $\mathcal{S}^* = \{i^*\}$.

Remark 5 Combining Lemmas 4 and 5, it can be seen that for small stubbornness θ, the influencer (i.e., agent 0) cares more about the long-term behaviour of random walks over $\mathcal{G}$, which is reflected in (11); while for large stubbornness, the influencer only considers the one-hop structure of the graph, encoded by the g_i's. This can also be explained from the mathematical form of the cost function $f(i;\theta,\omega)$: if we expand the right-hand term of (9), the weights assigned to high-order terms (i.e., $(1-\theta)^k$ for $k \geq 3$), which corresponds to multi-hop structures of $\mathcal{G}$, decreases rapidly as θ grows from 0 to 1.

C. Algorithm design

This subsection is going to provide an algorithm that generates an approximated solution to the problem (6).

We write P and $\mathbf{p}_0$ as $P(\mathcal{S})$ and $\mathbf{p}_0(\mathcal{S})$ to emphasize their dependence on $\mathcal{S}$. The first problem is to prove that each entry of $\mathrm{sp}_0(\mathcal{S})$ is monotonically increasing with $\mathcal{S}$.

Proposition 1 *Let Assumption 1 hold. For any* $\mathcal{S} \subset \mathcal{V}$ *and* $i \notin \mathcal{S}$*, it holds* $\mathrm{sp}_0(\mathcal{S} \cup \{i\}) > \mathrm{sp}_0(\mathcal{S})$.

The following proposition gives submodularity of $\mathrm{sp}_0(\mathcal{S})$.

Proposition 2 *Let Assumption 1 hold. The social power* sp_0 *is a submodular function of* $\mathcal{S}$.

With Propositions 1 and 2, a greedy algorithm can be designed to find an approximated solution to the problem (6), which is given as Algorithm 1. The following Theorem 1 is then obtained as a direct application of well-known properties of monotone submodular functions [21].

Algorithm 1 Greedy Algorithm for Social Power Maximization

Input: A strongly connect graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$; maximum number of new links K; a diagonal matrix of stubbornness Θ; link weight ω

Output: A subset of nodes $\mathcal{S} \subset \mathcal{V}$ with $|\mathcal{S}| = K$

```
1: Initialization: S ← ∅, S^c ← V\S
2: for k = 1 to K do
3:     sp_max ← 0, i_max ← 0
4:     for i = 1 to n do
5:         Compute sp_0(S ∪ {i}) by using Eqs. (4) and (5)
6:         if sp_0(S ∪ {i}) > sp_max then
7:             sp_max ← sp_0(S ∪ {i}), i_max ← i
8:         end if
9:     end for
10:    Update: S ← S ∪ {i_max}
11: end for
```

Theorem 1 *Consider the problem* (6) *for the graph* $\mathcal{G}$*. Let* $\mathcal{S}^*$ *be a solution and* $\mathcal{S}^{gd}$ *be the output of Algorithm 1. It then holds* $\mathrm{sp}_0(\mathcal{S}^{gd}) \geq (1 - \frac{1}{e})\mathrm{sp}_0(\mathcal{S}^*)$.

Note that in Algorithm 1, the computation of each $\mathrm{sp}_0(\mathcal{S} \cup \{i\})$ (in Step 5) requires the inverse of $I - (I - \omega \sum_{j \in \mathcal{S}} \mathbf{e}_j \mathbf{e}_j^\top)(I - \Theta)W$, for which th complexity is $O(n^3)$. Therefore, the computational complexity of Algorithm 1 is $O(K(n - \frac{K-1}{2})n^3)$.

The computation of matrix inverse can be impractical when the dimension n is large. To reduce computational complexity, we can make use of random walks generated by the Markov chain corresponding to the FJ model, as introduced in Section III-A, and compute an approximation of $\mathrm{sp}_0(\mathcal{S})$ by the following algorithm.

Algorithm 2 Approximation of $\mathrm{sp}_0(\mathcal{S})$

Input: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$; $\mathcal{S}$; Θ; ω; number and length of random walks r, ℓ

Output: An approximation of the absorbing probability of the state 0, $\mathrm{sp}_0^{\mathrm{apx}}(\mathcal{S})$

```
1: Initialization: sp_0^apx(S) ← 0
2: for k = 1 to r do
3:     Generate a random walk Y^(k) from the transition
       matrix (7) with length no larger than ℓ, and the starting
       state is uniformly distributed over {0, 1, ..., n}
4:     if the Y^(k) is absorbed by 0 then
5:         sp_0^apx(S) ← sp_0^apx(S) + 1
6:     end if
7: end for
8: Update: sp_0^apx(S) ← sp_0^apx(S)/r
```

For Algorithm 2, a natural question is how large r and ℓ are required to guarantee that $\mathrm{sp}_0^{\mathrm{apx}}(\mathcal{S})$ is close to $\mathrm{sp}_0(\mathcal{S})$. For the cases that all the agents have positive stubbornness, a high probability bound of $|\mathrm{sp}_0^{\mathrm{apx}}(\mathcal{S}) - \mathrm{sp}_0(\mathcal{S})|$ can be given as follows, in which

$$\mathrm{sp}_0^{\ell}(\mathcal{S}) := \frac{1}{n+1}\mathbf{1}^\top \sum_{k=0}^{\ell} H(\mathcal{S})^k (I - \Theta)\omega \sum_{j \in \mathcal{S}} \mathbf{e}_j + \frac{1}{n+1}.$$

Theorem 2 *Let Assumption 1 hold. Suppose that* $\theta_i \geq \theta > 0$ *hold for all* $i \in \mathcal{V}$*. In Algorithm 2, given* $\epsilon, \delta \in (0, 1)$ *and* $\sigma \in (0, 1)$*, let*

$$r \geq \frac{3}{\sigma^2 \epsilon^2 \mathrm{sp}_0^{\ell}(\mathcal{S})} \log(\frac{2}{\delta}),$$
$$\ell \geq \frac{\log((n+1)\theta(1-\sigma)\epsilon \mathrm{sp}_0^{\ell}(\mathcal{S})) - \log n\omega}{\log(1-\theta)} - 2.$$

Then, for any $\mathcal{S}$ *with* $|\mathcal{S}| \geq 1$*, it holds*

$$\mathbb{P}(|\mathrm{sp}_0^{apx}(\mathcal{S}) - \mathrm{sp}_0(\mathcal{S})| < \epsilon \mathrm{sp}_0^{\ell}(\mathcal{S})) \geq 1 - \delta.$$

Remark 6 From the definition of $\mathrm{sp}_0^{\ell}(\mathcal{S})$, it can be seen that $\mathrm{sp}_0^{\ell}(\mathcal{S}) \geq \frac{n\omega(1-\theta_{\max})+1}{n+1}$, where $\theta_{\max} := \max_{i \in \mathcal{V}}\{\theta_i | \theta_i < 1\} < 1$ (we only need to consider that for any $i \in \mathcal{S}$, it holds $\theta_i < 1$). This means that in Theorem 2, it should be $r = O(n)$ and $\ell = O(\log(n))$. Moreover, the complexity of one-step random walk is $O(d_{\max})$, with $d_{\max}$ the maximum degree of graph $\mathcal{G}$. Therefore, the complexity of Algorithm 2 is $O(n^2 \log(n))$, which indicates that if n is large, Algorithm 2 can reduce the computational complexity and obtain value of $\mathrm{sp}_0(\mathcal{S})$ in a relatively high accuracy.

The effectiveness of the greedy algorithm, i.e., Algorithm 1, is shown in the following example.

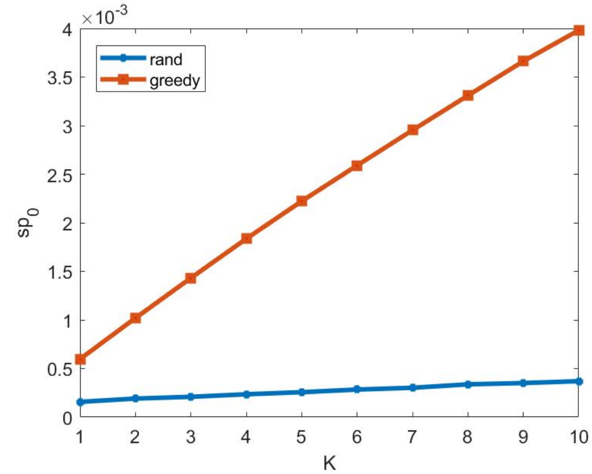

Fig. 1. Simulation for Example 1. Blue and red curve corresponds to random selection of $\mathcal{S}$ and Algorithm 1, respectively.

Example 1 Consider the social power optimization problem (6) on the LastFM Asia Social Network [23], in which $\mathcal{V} = 7624$ and $\mathcal{E} = 27,806$. The weight matrix W is obtained by normalize the adjacency matrix of the network. For each $i \in \mathcal{V}$, θ_i is randomly chosen from $[0.1, 1]$. Let $\omega = 0.2$. Such a large network size makes it impractical to implement an exhaustive search for the accurate solution of (6). For each

$K = 1, 2, \ldots, 10$, we run two algorithms to select $\mathcal{S}$ and compute the social power of agent 0 by using Algorithm 1 or randomly select K agents from $\mathcal{V}$ to consist $\mathcal{S}$, and repeat this process for 100 times to obtain an average value of sp_0. The simulation results is shown in Fig. 1. It can be seen that for each K, the social power generated by Algorithm 1 is obviously higher than that of the random selection.

IV. Conclusion

In this paper, we studied the problem that an external influencer optimizes her social power in an FJ model, by selecting a given number of agents to add links. This problem was shown to be equivalent to maximizing the absorbing probability to the influencer in the augmented Markov chain. The cases of small and large stubbornness were studied analytically to explore how the stubbornness and link weight affect the solution of the problem. Moreover, the social power the influencer was proved to be monotone and submodular with the selected agent set, and computationally, a greedy algorithm was proposed to generate a suboptimal solution. For large network size, random walks were used to reduce the complexity of computing matrix inverse, which was required in the greedy algorithm.

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