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A Smooth Bounded Confidence Model Maintaining Clustering Phenomenon

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Abstract: In this paper, we propose a stochastic bounded confidence model. At every time slot, each individual receives an external social signal which is the average of opinions of its neighbors. Then agents compare the signals with their own personal biases which are defined as the initial values that ones hold. With positive probability, the agents either accept the opinion or persuade themselves with the help of personal prejudices. The probability of acceptance is reversely proportional to the opinion discrepancy between the signal and the bias. The model is modified as a continuous opinions discrete actions (CODA) model and thus is a Markov chain taking values on a finite state space. It is verified that the chain is aperiodic and finally converges in distribution to some invariant measure. The classification of states shows that the influences of distant opinions will boost consensus while the presence of personal biases promote clustering. The model also combines DeGroot model with Friedkin-Johnson model as well, by using a bounded confidence framework.

Key Words: social networks, opinion dynamics, bounded confidence model, Friedkin-Johnson model, DeGroot model

1 Introduction

Opinion dynamics is a booming field endeavoring to model and interpret processes of social influence and attitude change with the help of mathematical or simulation methods [1, 2]. Because of potential applications in economics, social sciences and management [3–5], study on opinion formation has attracted attention of researchers from a variety of domains.

Some investigators believe that the understanding of interpersonal influences is the first step to answer the community cleavage problem which is one of the most crucial conundrums of social sciences [3, 6]. This problem is also known as Abelson's diversity puzzle [7], i.e., what assumptions are fundamental for models to generate not only agreements but also cleavage phenomena of opinion.

Opinion dynamics models can be mainly categorized into three classes according to their social psychological backgrounds [1]. Assimilative influence models presuppose social structural topologies of individuals. Agents exchange views with their "neighbors", namely, people who have direct contact with the former. French-DeGroot model [8, 9] is a classic example where individuals update their opinions as an average of neighbors' views. For DeGroot model, the group will reach a consensus if and only if the network is quasi-strongly connected [2]. However, the clustering behavior can only be generated when there are multiple strongly connected components, which is not a satisfactory solution to Abelson's puzzle. A successful modification, which is called Friedkin-Johnson model in the literature, is to consider the group's history [2, 3]. In other words, individuals take their initial views that represent their personal interest into account when updating.

The second category are bounded confidence models, which are also referred to as models with similarity bias [1]. In such models, e.g., Deffuant-Weisbuch model [10] and Hegselmann-Krause model [11], the similarity of people determines whether they will influence each other or not. The major assumption is that people who hold close opinions feel like each other, which is called homophily [12]. Because of the simple assumptions, they have been widely studied in different fields [1, 2, 6]. But these models have several deficiencies, which will be discussed later.

Group polarization [13] is a special phenomenon in social psychology and cannot be predicted by neither of the above two classes of models. And thus belief evolution models over signed networks [1, 14, 15] are proposed and well investigated. Here individuals keep their opinions away from each other when they have repulsive connections. However, there is few empirical evidence for repulsive influence [1] and there are models without negative links which can also generate polarization phenomenon [16]. For other kinds of opinion dynamics models, one may refer to [1, 6].

It is worth noting that agents in bounded confidence models always ignore distant views, which has been widely criticized [6, 16]. As a matter of fact, the behavior pattern will change drastically when releasing this assumption. Specifically speaking, the existence of only a tiny probability that individuals can be influenced by faraway views will lead to a significant decrease of opinion clusters [6]. This modification of update rule is called a smooth confidence bound [6, 17, 18]. Another problem is the clustering robustness clustering of bounded confidence models. It is verified that fragmentation of Hegselmann-Krause model will vanish when arbitrarily small noise exists, and endogenous differences, e.g., personal bias or group's history, may account for the clustering phenomenon [19].

Therefore, we propose a new model in this paper considering smooth confidence bounds, influences of distant opinions and individual prejudices. Although there are bounded confidence models with prejudices [20, 21], they do not include the effects of remote views. In our model, individuals are allowed to be influenced by distinct opinions through a

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predefined structural topology. The model is put forward in a stochastic bounded confidence framework and agents can either accept a discussion outcome or persuade themselves with their biases with positive probabilities.

Our novel model establishes a connection between assimilative influence models and similarity bias models. More specifically, when the confidence tendency function is heavy tailed, the behavior of the model will be more similar to a discrete-state DeGroot model. On the other hand, when the tail of the tendency function is thin, opinion clustering, which is a common phenomenon of Friedkin-Johnson model, will take place with high probability. From this point of view, the proposed model partly combines the DeGroot model and Friedkin-Johnson model using a bounded confidence framework. The stochastic stability of the model is obtained and the result also confirms that social influences of distant views boost consensus while the existence of personal bias promotes clustering. Unlike bounded confidence models, our model's behavior won't change a lot when random noise is added to the system.

The rest of the paper is arranged as follows. In section 2, some preliminaries and notations are given and our novel model will be proposed in section 3. Main results can be found in section 4 and numerical simulations are in section 5. Section 6 concludes the paper.

2 Preliminaries and notations

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represent a simple graph, where $\mathcal{V}$ is the set of agents and $\mathcal{E}$ is the set of edges between pairs of agents. A sequence of consecutive edges $\{(i, k_1), \cdots, (k_l, j)\}$ is called a path between i and j. We say that i and j are connected if there is a path between them. A graph is referred to as connected if every pair of agents is connected. Denote $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ to be the neighbor set of i. Let W be the adjacent matrix of $\mathcal{G}$, that is, $W_{ij} = \frac{1}{|\mathcal{N}_i|}$ if $(i, j) \in \mathcal{E}$ and $W_{ij} = 0$ otherwise.

Define the length of the shortest path between i and j to be the distance $d(i, j)$ between these two agents. Define the diameter of a connected graph $\mathcal{G}$ as $d(\mathcal{G}) := \max_{i,j\in\mathcal{V}}\{d(i, j)\}$, that is, the longest distance in $\mathcal{G}$.

A stochastic process $\{X(t), t \geq 0\}$, taking values on a countable state space $\mathcal{X}$, is said to be a Markov chain if $\mathbb{P}\{X(t) = \boldsymbol{x}_t | X(t-1) = \boldsymbol{x}_{t-1}, \cdots, X(0) = \boldsymbol{x}_0\} = \mathbb{P}\{X(t) = \boldsymbol{x}_t | X(t-1) = \boldsymbol{x}_{t-1}\}$, where $t \geq 1$ and $\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_t \in \mathcal{X}$. The process $\{X(t)\}$ is a time-homogeneous Markov chain if the probability $\mathbb{P}\{X(t+1) = \boldsymbol{x}_1 | X(t) = \boldsymbol{x}_0\}$ depend only on the values of $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$, and are independent of the time slot. For a time-homogeneous Markov chain, we write $\mathrm{P}(\boldsymbol{x}, \boldsymbol{y}) = \mathbb{P}\{X(1) = \boldsymbol{y} | X(0) = \boldsymbol{x}\}$ as the transition probability from state $\boldsymbol{x}$ to state $\boldsymbol{y}$ and $\mathrm{P}^t(\boldsymbol{x}, \boldsymbol{y}) = \mathbb{P}\{X(t) = \boldsymbol{y} | X(0) = \boldsymbol{x}\}$ as the t-step transition probability from state $\boldsymbol{x}$ to state $\boldsymbol{y}$.

If there exists $t \geq 1$ such that $\mathrm{P}^t(\boldsymbol{x}, \boldsymbol{y}) > 0$ we say that $\boldsymbol{y}$ is reachable from $\boldsymbol{x}$. For a state $\boldsymbol{x} \in \mathcal{X}$ define $\tau_{\boldsymbol{x}} = \inf\{t > 0 : X(t) = \boldsymbol{x}\}$ to be the first return time to $\boldsymbol{x}$. The state $\boldsymbol{x}$ is said to be recurrent if $\mathbb{P}\{\tau_{\boldsymbol{x}} < \infty | X(0) = \boldsymbol{x}\} = 1$. We call a state $\boldsymbol{x}$ is positive recurrent if $\mathbb{E}_{\boldsymbol{x}}\{\tau_{\boldsymbol{x}}\} < \infty$.

In the following sections, we use bold lower case letters to denote vectors, for example, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$. Let $\mathbf{1}$ be the vector whose all entries are 1. Define rounding functions $\lceil x \rceil = \min\{y \in \mathbb{Z} : x \leq y\}$ and $\lfloor x \rfloor = \max\{y \in \mathbb{Z} : y \leq x\}$ for $x \in \mathbb{R}$. For a matrix A, denote its i-th row as $A^{(i)}$.

3 Problem formulation

We propose a stochastic bounded confidence model [17] and restrict it to dynamics in $\mathbb{Z}$ to avoid technicalities. In fact, there are a variety of models assuming the opinion is nominal or discrete [1], e.g., voter model [22]. A discrete belief may represent a choice from several options or a rating for a movie, product, etc [17]. Our model can also be regarded as a continuous opinions and discrete actions (CODA) model where discrete actions (or views) are revealed and observed but everyone carries a continuous opinion indeed [23].

Let $\mathcal{V} = \{1, 2, \ldots, n\}$ be the set of agents. Hence graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ define the underlying social network. Denote the opinion of agent i at time t as $x_i(t)$ and let $\boldsymbol{x}(t)$ be the opinion vector at time t. At each time, agents in $\mathcal{V}$ update their states independently of each other and of previous updates. To be specific, agent i follows the rule below.

$$X_i(t+1) = \begin{cases} r(S_i(t)) \\ \qquad \text{with probability } f\big(|S_i(t) - u_i|\big), \\ r\big((1-h)S_i(t) + hu_i\big) \\ \qquad \text{with probability } 1 - f\big(|S_i(t) - u_i|\big), \end{cases} \tag{1}$$

where $S_i(t) = \sum_{j\in\mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} x_j(t)$ is the external social signal; $h \in [0, 1]$ is the confidence index of agents; $r(\cdot)$ is a random rounding function with $\mathbb{P}\{r(x) = \lceil x \rceil\} = \mathbb{P}\{r(x) = \lfloor x \rfloor\} = \frac{1}{2}$ for $x \in \mathbb{R}$; $f(\cdot) : \mathbb{R}^+ \cup \{0\} \to [0, 1]$ is a nonincreasing tendency function and $0 < f(x) < 1$ when $x > 0$; and $\{u_i, 1 \leq i \leq n\}$ is a set of fixed integers representing personal biases.

The probabilities with which the rounding function $r(x)$ takes ceiling and flooring respectively can be defined to be related to x but there is no difference as long as both of them have positive probabilities. The real function $f(\cdot)$ in (1) can be considered as one's tendency to accept others' opinion [6, 17], since the smaller $|S_i(t) - u_i|$ is, the larger the probability for an agent to accept its neighbors' views becomes. On the contrary, if $|S_i(t) - u_i|$ is large, it is more likely that it will persuade itself to maintain its own standpoint (u_i for agent i).

Intuitively, after a discussion, every individual has to decide whether to accept the result or not. If the result is close to one's priori prejudice, then the agent will be more likely to take the belief as its own. However, if not, it will modify the opinion with its bias. Here one can find two differences from the classic bounded confidence models. One is that if an agent decides not to accept an opinion, it will be partly influenced by the opinion rather than adhere to its own. This update rule is not uncommon [20, 21]. The other is that in classic models, the basis for judging similarity is the current state $X_i(t)$ rather than u_i. But since the introduction of individual prejudice, it is more reasonable that personal bias could have a greater impact on an agent.

Note that once $\{u_i\}$ is selected, $\{X(t)\}$ is bounded. Thus denote $\overline{u} := \max_{1\le i\le n} u_i$, $\underline{u} := \min_{1\le i\le n} u_i$ and $\Delta u := \overline{u} - \underline{u}$, without loss of generality, one could assume that $\underline{u} \le X_i(0) \le \overline{u}$, $1 \le i \le n$. We further denote $\mathbb{Z}_u := \mathbb{Z} \cap [\underline{u}, \overline{u}]$ and thus $X(t) \in (\mathbb{Z}_u)^n$, $t \ge 0$.

The model (1) is also related to two models in the literature, i.e., DeGroot model [8] and Friedkin-Johnson model [3]. It is easy to see that the first kind of update rule in (1) is a homogeneous DeGroot model:

$$\boldsymbol{x}(t+1) = W\boldsymbol{x}(t), \quad t \ge 0, \tag{2}$$

and the second is a homogeneous F-J model:

$$\boldsymbol{x}(t+1) = \Lambda W \boldsymbol{x}(t) + (I_n - \Lambda)\boldsymbol{u}, \tag{3}$$

where $\boldsymbol{u} = x(0)$ and here $\Lambda = (1-h)I_n$. The difference is that both update rules in (1) are in discrete state space.

4 Main Results

4.1 Classification of states

From section 3, we know that (1) is a discrete-time Markov chain with finite states. So the first step is to categorize the states of (1). Since the update rule of our model is related to DeGroot and Friedkin-Johnson model, the equilibriums of the latter two models may play a crucial rule in our analysis. From now on, we assume that $\{u_i\}$ is fixed and $\overline{u} - \underline{u} \ge 1$, therefore every conditions in (1) takes place with probability at least $\frac{1}{2}[f(\overline{u}-\underline{u}) \wedge (1-f(0))] > 0$ (Here there is only one exception, that is, $|S_i(t) - u_i| = 0$. But when this happens, the results of both update rules are the same).

The following Lemma shows that states $c\mathbf{1}$ are aperiodic. Intuitively, it means that when a crowd has reached a consensus, the situation can last for several steps with positive probability.

Lemma 1. $\mathrm{P}(c\mathbf{1}, c\mathbf{1}) > 0$, $\forall c \in \mathbb{Z}_u$.

Proof. Suppose $X(0) = c\mathbf{1}$, for some $c \in \mathbb{Z}_u$, then $S_i(0) = c$, $\forall i \in \mathcal{V}$. Thus with probability $f(|c-u_i|) > 0$, $X_i(1) = r(S_i(0)) = r(c) = c$, $\forall i \in \mathcal{V}$, which implies that $\mathrm{P}(c\mathbf{1}, c\mathbf{1}) > 0$. $\square$

Lemma 2. *For any state $\boldsymbol{x} \in (\mathbb{Z}_u)^n$, $\exists t > 0$, such that $\mathrm{P}^t(\boldsymbol{x}, c\mathbf{1}) > 0$, for some $c \in \mathbb{Z}_u$.*

Proof. Let $S_{\max}(t) := \max_{1\le i\le n} S_i(t)$, $S_{\min}(t) := \min_{1\le i\le n} S_i(t)$ and $\mathscr{X}(t) := \max_{1\le i\le n} X_i(t) - \min_{1\le i\le n} X_i(t)$. Suppose $X(0) = \boldsymbol{x}$ and $\mathscr{X}(0) \ge 1$.

If $S_{\max}(0) - S_{\min}(0) \le 1$, then $\mathrm{P}(\boldsymbol{x}, \lfloor S_{\max}(0) \rfloor \mathbf{1}) > 0$.

Provided that $S_{\max}(0) - S_{\min}(0) > 1$, and either $S_{\max}(0)$ or $S_{\min}(0)$ is not an integer, then with positive probability, $\mathscr{X}(1) = \lfloor S_{\max}(0) \rfloor - \lceil S_{\min}(0) \rceil < S_{\max}(0) - S_{\min}(0) \le \mathscr{X}(0)$.

If rather, $S_{\max}(0) - S_{\min}(0) > 1$ and both $S_{\max}(0)$ and $S_{\min}(0)$ are integers, then with positive probability, $X_i(1) = \lfloor S_i(0) \rfloor$, $\forall i \in \mathcal{V}$. For such states, let $\overline{\mathcal{X}}(1) := \{i \in \mathcal{V} : X_i(1) = S_{\max}(0)\}$ and $\underline{\mathcal{X}}(1) := \{i \in \mathcal{V} : X_i(1) = S_{\min}(0)\}$, we have that $\exists l \in \overline{\mathcal{X}}(1)$, $\exists k \in \mathcal{N}_l$ such that $X_k(1) < S_{\max}(0)$. Otherwise, $\forall l \in \overline{\mathcal{X}}(1)$, $\forall k \in \mathcal{N}_l$, $X_k(1) = S_{\max}(0)$ holds, which leads to $\mathcal{V} = \overline{\mathcal{X}}(1)$. That is because $\mathcal{G}$ is connected and $\overline{\mathcal{X}}(1) \ne \emptyset$. This contradicts with $\underline{\mathcal{X}}(1) \ne \emptyset$. Thus,

$$S_l(1) = \frac{1}{|\mathcal{N}_l|}\sum_{j\in\mathcal{N}_l} x_j(1) < \frac{1}{|\mathcal{N}_l|}\sum_{j\in\mathcal{N}_l} S_{\max}(0) = S_{\max}(0).$$

Therefore, with positive probability, $X_i(2) = \lfloor S_i(1) \rfloor$ and at the same time, $X_l(2) < S_{\max}(0)$ for all $l \in \overline{\mathcal{X}}(1)$ with some $k \in \mathcal{N}_l$ such that $X_k(1) < S_{\max}(0)$. Repeat this procedure and we have $X_i(d(\mathcal{G})+1) < S_{\max}(0)$, $\forall i \in \mathcal{V}$, with positive probability, where $d(\mathcal{G})$ is the diameter of graph $\mathcal{G}$. Hence, $\mathbb{P}\{\mathscr{X}(d(\mathcal{G})+1) < \mathscr{X}(0) | X(0) = \boldsymbol{x}\} > 0$, that is, $\mathrm{P}^{d(\mathcal{G})+1}(\boldsymbol{x}, \boldsymbol{z}) > 0$ for some $\boldsymbol{z} \in (\mathbb{Z}_u)^n$ such that $\max\{z_i\} - \min\{z_i\} < \mathscr{X}(0)$.

Since $\mathscr{X}(t)$ are integers, by Lemma 1, we have $\mathrm{P}^{t_0}(\boldsymbol{x}, c\mathbf{1}) > 0$ for some $c \in \mathbb{Z}_u$ and $t_0 = (d(\mathcal{G})+1)\mathscr{X}(0)$. $\square$

Lemma 2 shows that whatever state the model (1) starts with, there is a positive probability that the system reaches a consensus for finite time. Note that this may not be true for continuous opinion models. The next Lemma tells us that, if h is small enough, then the states in the consensus vector set $\{c\mathbf{1} : c \in \mathbb{Z}_u\}$ are reachable from each other, which makes itself play a crucial role in classifying states of (1).

Lemma 3. *Assume that $h \le \frac{1}{\Delta u}$, then the states in the set $\{c\mathbf{1} : c \in \mathbb{Z}_u\}$ are reachable from each other.*

Proof. It suffices to verify that $(\underline{u}+1)\mathbf{1}$ is reachable from $\underline{u}\mathbf{1}$. Let $X(0) = \underline{u}\mathbf{1}$ and $l \in \mathcal{V}$ such that $u_l = \overline{u}$. Then with positive probability, $X_l(1) = \lceil (1-h)S_l(0) + h\overline{u} \rceil = \lceil \underline{u} + h(\Delta u) \rceil = \underline{u} + 1$, while $X_k(1) = \lceil S_k(0) \rceil = u_m$, $k \ne l$. Further we have $\mathbb{P}\{X_k(2) = \lceil S_k(1) \rceil | X(1) = \boldsymbol{x}\} = \mathbb{P}\{X_k(2) = \underline{u}+1 | X(1) = \boldsymbol{x}\} \ge \frac{1}{2} f(|W^k \boldsymbol{x} - u_k|)$, $\forall k \in \mathcal{N}_l$. Thus recursively, by Lemma 1, $\mathrm{P}^{t_0}(\underline{u}\mathbf{1}, (\underline{u}+1)\mathbf{1}) > 0$, where $t_0 = d(\mathcal{G}) + 1$. $\square$

Intuitively, this lemma means that if every agent is not so stubborn, then it is possible for the chain to reach every consensus state. But, in fact, this phenomenon happans because of the existence of personal prejudice.

For Markov chains with finite states, there must exist some positive recurrent class and all recurrent states are positive [24]. From Lemma 2, we know that for every positive recurrent class $\mathcal{R}$ of (1), there is some $c \in \mathcal{Z}_u$ such that $c\mathbf{1} \in \mathcal{R}$. Moreover, by Lemma 1, such recurrent class is aperiodic. Thus the following theorem holds:

Theorem 1. *For fixed $\{u_i\}$, the Markov chain* (1) *has at least one positive recurrent class and all positive states are aperiodic. Moreover, if $h \le \frac{1}{\Delta u}$, then* (1) *has only one positive recurrent class.*

Now we discuss the classification of the limit point of Friedkin-Johnson model (3). When W is strongly connected, we know that $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{u}^*$, where $\boldsymbol{u}^* = (I_n - \Lambda W)^{-1}(I_n - \Lambda)\boldsymbol{u}$. But the entries of $\boldsymbol{u}^*$ need not be integers, so let $\mathcal{U} := \{\boldsymbol{x} \in \mathbb{Z}_u : x_i = \lceil u_i^* \rceil \text{ or } \lfloor u_i^* \rfloor\}$ and we have

Proposition 1. *For any state $\boldsymbol{x} \in (\mathbb{Z}_u)^n$, $\exists t > 0$, such that $\mathrm{P}^t(\boldsymbol{x}, \boldsymbol{z}) > 0$ for some $\boldsymbol{z} \in \mathcal{U}$.*

Proof. Denote $\tilde{X}_i(t+1) = (1-h)S_i(t) + hu_i$ and $\mathscr{M}(\boldsymbol{y}) := \max_{1\le i\le n} |y_i - u_i^*|$, $i \in \mathcal{V}$. For fixed initial value $X(0) = \boldsymbol{x}$, because

$$\tilde{X}(1) - u^* = \Lambda W(X(0) - u^*),$$

we have that

$$|\tilde{X}_i(1) - u_i^*| \le (1-h)\mathscr{M}(X(0)).$$

If $|\tilde{X}_i(1) - u_i^*| < 1$, since

$$X_i(1) \le \lceil \tilde{X}_i(1) \rceil < \tilde{X}_i(1) + 1 < u_i^* + 2;$$

$$X_i(1) \ge \lfloor \tilde{X}_i(1) \rfloor > \tilde{X}_i(1) - 1 > u_i^* - 2,$$

we know that $|X_i(1) - u_i^*| \le 1$. On the other hand, if $|\tilde{X}_i(1) - u_i^*| \ge 1$, then with positive probability, $|X_i(1) - u_i^*| \le |\tilde{X}_i(1) - u_i^*|$. As a result, $\mathrm{P}(\boldsymbol{x}, \boldsymbol{y}) \ge p^* > 0$, where p^* is a constant independent of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{x}$, $\boldsymbol{y}$ are such that $\mathscr{M}(\boldsymbol{y}) \le 1 \vee (1-h)\mathscr{M}(\boldsymbol{x})$. Therefore, there exists large enough $t_0 > 0$ and some state $\boldsymbol{z} \in (\mathbb{Z}_u)^n$ satisfying $\mathrm{P}^{t_0}(\boldsymbol{x}, \boldsymbol{z}) > 0$ with $\mathscr{M}(\boldsymbol{z}) \le 1$, i.e., $|z_i - u_i^*| \le 1, \forall i \in \mathcal{V}$. Similar to (4.1), we have $\mathrm{P}(\boldsymbol{z}, \tilde{\boldsymbol{u}}) > 0$ for some $\tilde{\boldsymbol{u}} \in \mathcal{U}$, which completes the proof. □

One of the important properties of our concern is the model's robustness to noise. But as we can see above, the reachability of the consensus vector set $\{c\mathbf{1} : c \in \mathbb{Z}_u\}$ and the set $\mathcal{U} := \{\boldsymbol{x} \in \mathbb{Z}_u : x_i = \lceil u_i^* \rceil \text{ or } \lfloor u_i^* \rfloor\}$ is guaranteed by the formulation of the model. Suppose that the update of agents in (1) is added with some independent noise $n_i(t)$ taking values in $[-\Delta u, \Delta u]$ and $\mathbb{P}\{n_i(t) = 0\} > 0$. And further assume the chain is bounded in $[\underline{u}, \overline{u}]$, then it is not hard to see that the conclusions above still hold.

To end this section, a few examples are given to show that actually not all states in $(\mathbb{Z}_u)^n$ are positive when Δu is large enough:

Proposition 2. *Suppose $\Delta u \ge 2n$. If there exists agents i and j with $d(i,j) \le 2$, $u_i < \overline{u}$ and $u_j > \underline{u}$, then states in set $\mathcal{S} := \{\boldsymbol{x} \in (\mathbb{Z}_u)^n : x_i = \overline{u}, x_j = \underline{u}\}$ are not reachable from any other states in $(\mathbb{Z}_u)^n$.*

Proof. Provided that for some time $t > 0$, $X(t)$ reaches set $\mathcal{S}$ at the first time, then $X_i(t-1) \ne \overline{u}$ or $X_j(t-1) \ne \underline{u}$. From the update rule (1) and the assumptions, $S_i(t-1) \ge \overline{u} - 1$, $S_j(t-1) \le \underline{u} + 1$. Thus, $X_k(t-1) \ge \overline{u} - n$, $X_l(t-1) \le \underline{u} + n$, $k \in \mathcal{N}_i$, $l \in \mathcal{N}_j$ and the equalities cannot hold at the same time. Since $d(i,j) \le 2$, i and j share a neighbor m satisfying $\overline{u} - n \le X_m(t-1) < \underline{u} + n$ or $\overline{u} - n < X_m(t-1) \le \underline{u} + n$, which is impossible because from the assumption, $\overline{u} - n \ge \underline{u} + n$. □

Proposition 3. *Suppose $\Delta u > 2$ and $\mathcal{G}$ is complete graph. If there exists agents i and j with $u_i < \overline{u}$ and $u_j > \underline{u}$, then states in set $\mathcal{S} := \{\boldsymbol{x} \in (\mathcal{Z}_u)^n : x_i = \overline{u}, x_j = \underline{u}\}$ are not reachable from any other states in $(\mathcal{Z}_u)^n$.*

Proof. Following the notations in the proof of Proposition 2, we have that $S_i(t-1) \ge \overline{u} - 1$, $S_j(t-1) \le \underline{u} + 1$. Since $\mathcal{V} = \mathcal{N}_i, \forall i \in \mathcal{V}$, $\underline{u} + 1 < \overline{u} - 1 \le S_i(t-1) = S_j(t-1) \le \underline{u} + 1$, which is impossible. □

4.2 Convergence of the distribution and expectation

From Theorem 1, we have the following result on the stationary distribution of (1):

Theorem 2. *For fixed $\{u_i\}$, the Markov chain (1) converges in distribution to some invariant measure. If $h \le \dfrac{1}{\Delta u}$, then the invariant measure is unique.*

Corollary 1. *For fixed $\{u_i\}$ and fixed initial value $X(0) = \boldsymbol{u}$, the model (1) converges in distribution to an invariant measure.*

The behavior of Markov chain (1) can be partly illustrated by the expectation, and it is shown that the limit of expectation depends on the tendency function f and the limit distribution.

Proposition 4. $\lim_{t\to\infty} \mathbb{E}\{X(t)\} = \sum_{\boldsymbol{x}\in(\mathbb{Z}_u)^n} \boldsymbol{g}(\boldsymbol{x})\mu(\boldsymbol{x})$, *where $\boldsymbol{g}(\boldsymbol{x})$ is a deterministic vector function dependent only on $\boldsymbol{x}$, and μ is an invariant measure of (1) which may depend on the initial distribution if $h > \dfrac{1}{\Delta u}$.*

Proof. From update rule (1), it follows that

$$\begin{aligned}
&\mathbb{E}\{X_i(t+1)\} \\
=& \sum_{y\in\mathbb{Z}_u} y\mathbb{P}\{X_i(t+1) = y\} \\
=& \sum_{y\in\mathbb{Z}_u}\sum_{\boldsymbol{x}\in(\mathbb{Z}_u)^n} y\mathbb{P}\{X_i(t+1) = y | X(t) = \boldsymbol{x}\}\mathbb{P}\{X(t) = \boldsymbol{x}\} \\
=& \sum_{\boldsymbol{x}\in(\mathbb{Z}_u)^n}\sum_{y\in\mathbb{Z}_u} y\mathbb{P}\{X_i(t+1) = y | X(t) = \boldsymbol{x}\}\mathbb{P}\{X(t) = \boldsymbol{x}\} \\
=& \sum_{\boldsymbol{x}\in(\mathbb{Z}_u)^n} \mathbb{P}\{X(t) = \boldsymbol{x}\}\cdot \\
&\quad \Big[\frac{1}{2} f(|W^{(i)}\boldsymbol{x} - u_i|)(\lceil W^{(i)}\boldsymbol{x}\rceil + \lfloor W^{(i)}\boldsymbol{x}\rfloor) + \\
&\quad \frac{1}{2}(1 - f(|W^{(i)}\boldsymbol{x} - u_i|))(\lceil (1-h)W^{(i)}\boldsymbol{x} + hu_i\rceil \\
&\quad + \lfloor (1-h)W^{(i)}\boldsymbol{x} + hu_i\rfloor)\Big] \\
:=& \sum_{\boldsymbol{x}\in(\mathbb{Z}_u)^n} g_i(\boldsymbol{x})\mathbb{P}\{X(t) = \boldsymbol{x}\},
\end{aligned}$$

It follows from Theorem 2 that $\mathbb{P}\{X(t) = \boldsymbol{x}\} \to \mu(\boldsymbol{x})$, as $t \to \infty$, where μ is an invariant measure of (1) which may depend on the initial distribution if $h > \dfrac{1}{\Delta u}$. Whence the conclusion follows. □

From Proposition 4, we know that the heavier tailed the tendency function f is, the more similar the limit expectation behaves to the corresponding DeGroot model. On the contrary, thin tailed tendency f leads to F-J model-like behavior. This phenomenon is also illustrated in the next section.

5 Numerical Simulations

In this section, we run several numerical simulations to demonstrate results in the previous section. Firstly, to show the reachability of consensus vector set $\{c\mathbf{1} : c \in \mathbb{Z}_u\}$, a sample path of a 3-agent network is showed in Fig. 1, where

$\overline{u} = 4$, $\underline{u} = 1$ and $h = 0.2 < \frac{1}{3}$. The aperiodicity of such states can also be found in this figure.

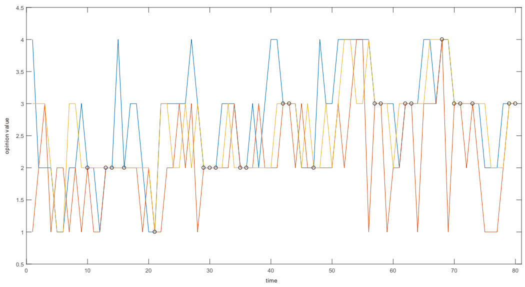


Fig. 1: Demonstration of the reachability and aperiodicity of $\{c\mathbf{1} : c \in \mathbb{Z}_u\}$. The black circles represent points in that set.

Next we show the relation between the behavior of the model (1) and the acceptance tendency function f. We use a 6-agent network whose topology is shown in Fig. 2 and let $\overline{u} = 4$, $\underline{u} = 1$ and $h = 0.2 < \frac{1}{3}$. For different probability of acceptance function, we choose $f_1(x) = \frac{1}{(x+1)^{0.1}}$, $f_2(x) = \frac{1}{x+1}$ and $f_3(x) = \frac{1}{(x+1)^{10}}$, whose graphs are in Fig. 3. Note that f_1 is heavy-tailed while f_3 is thin-tailed.

The approximations of the marginal stationary distributions of an agent i with prejudice $u_i = 4$ for f_1, f_2, f_3, $f_4 \equiv 0$ and $f_5 \equiv 1$ are shown in Fig. 4(a), 4(b), 4(c), 5(a) and 5(b) respectively. It is worth noting that when $f \equiv 0$, the model (1) is a discrete-state Friedkin-Johnson model while when $f \equiv 1$, the model (1) is a discrete-state DeGroot model.

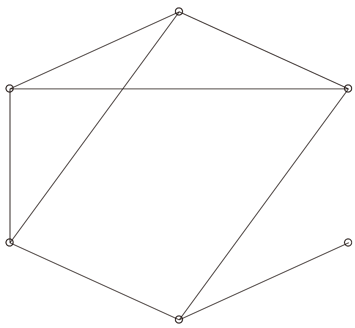


Fig. 2: A social network with 6 agents

Finally, the ergodic property of model (1) is shown in Fig. 6, that is, $\frac{1}{n}\sum_{t=1}^{n} X(t)$ converges a.s.

6 Conclusion

In this paper, we propose a opinion model with smooth confidence bound maintaining clustering phenomenon. The model established connections among bounded confidence models, DeGroot model and Friedkin-Johnson model. The stochastic stability of model (1) is obtained, which shows that for a homophily model, the influence of distant views may alter the behavior of the system a lot, but clustering can

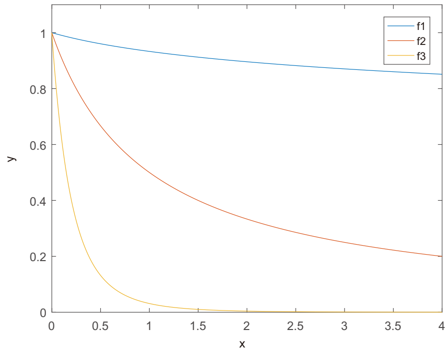


Fig. 3: The acceptance tendency functions

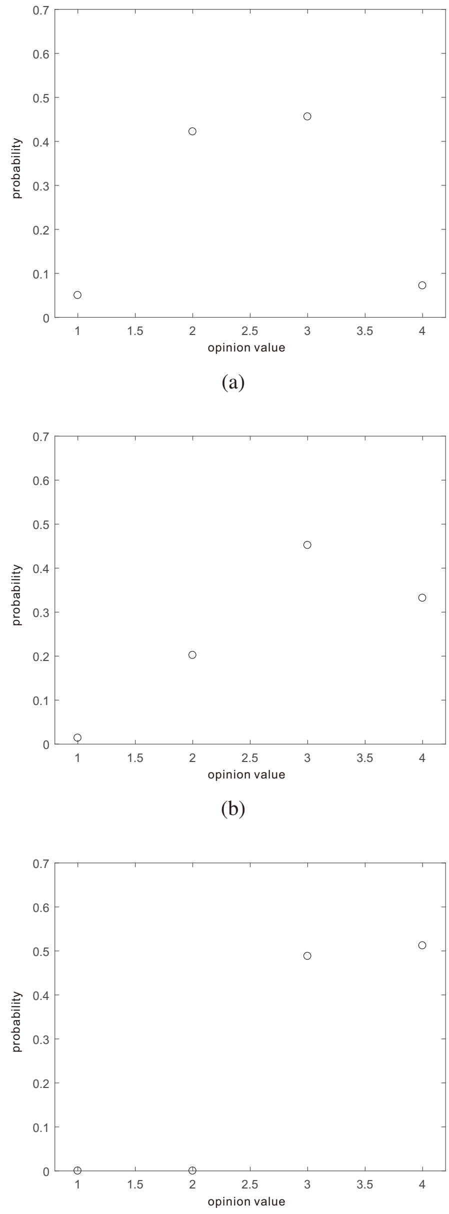
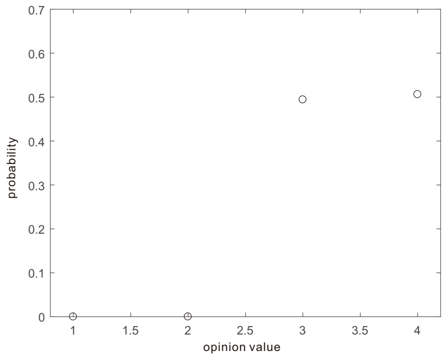
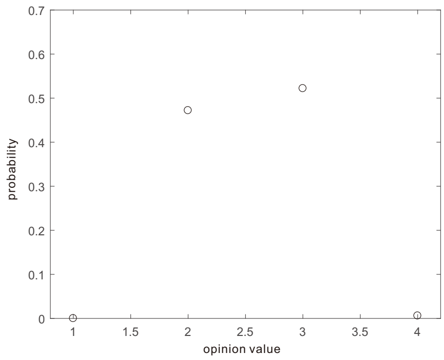


Fig. 4: The approximations of the marginal stationary distributions of an agent i with prejudice $u_i = 4$ for (a) f_1, (b) f_2, and (c) f_3.



(a)



(b)

Fig. 5: The approximations of the marginal stationary distributions of an agent i with prejudice $u_i = 4$ for (a) $f_4 \equiv 0$ and (b) $f_5 \equiv 1$

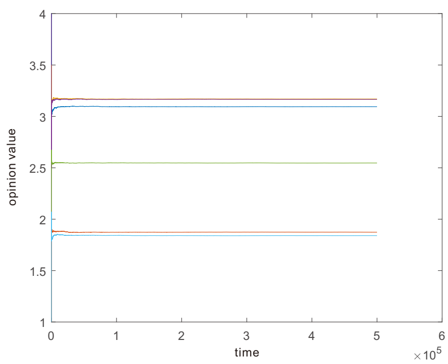


Fig. 6: The ergodic property of model (1).

still take place with the presence of personal bias. Further researches may focus on the effects of different tendency functions, detailed results for special kinds of graphs and applications to the real world opinion formation processes.

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