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A Random Opinion Formation Model over Signed Networks

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Abstract— **We study a random opinion dynamics model over signed networks with two different topological assumptions, that is, strongly connected and quasi-strongly connected. The model is proposed in a gossip algorithm form for simplicity but can be extended to a general one: at each time, a subgraph of the underlying interaction graph is selected and agents exchange their opinions based on this selected subgraph. It is shown that when the interacting graph is strongly connected, structurally balance is crucial for opinion clustering. However, when the interacting graph is quasi-strongly connected, the structurally balance assumption of the rooted graph is not enough for the convergence of the system and agents that are not roots may end in fluctuation.**

I. INTRODUCTION

In the past decades, a great deal of research has been conducted on network dynamics from diverse disciplines, and various of topics have been studied, for example, the evolvement of network structures, epidemics in networks, voting models and so on [6], [16].

Opinion dynamics, which attempts to figure out the consequences of opinion formation processes among a group of interacting agents [10], is also one of these booming fields. The research dates back to the well-known model proposed by DeGroot [4], where individuals update their opinions as convex combinations of their own and others' opinions and finally reach consensus. In fact, opinion-behavior dynamics is relate to the fundamental problem of sociology, i.e., "the coordination and control of social systems" [9]. Recent inquiries on protocols for consensus and synchronization in multi-agent networks [12], [19] also provide lots of mathematical models and tools for the study of opinion dynamics. For instance, gossip algorithms, which are commonly used in engineering [7], could also be used to describe social influence processes taking place in a random way [1].

However, in reality, it is more common that beliefs of individuals fail to reach consensus but end in disagreement, clustering [18], and even fluctuating [1]. This gives rise to the community cleavage problem, which is central to the coordination and control of social systems [9]: to reveal the mechanisms that fail to generate consensus.

At present there are several explanations for such phenomenon. The first is an opinion dynamics model with stubborn agents [9]: an individual is influenced by not only others but also its original opinion. The second is the class of confidence bound models, e.g. [10], in which only individuals whose opinions are close enough exchange their beliefs.

Another attempt is the introduction of signed networks [6] or antagonistic interactions [2]. The concepts of signed networks and structurally balanced theory were first brought in by Heider and further generalized by Cartwright and Harary to illustrate the positive and negative relationships among groups of people [3]. Indeed, the cooperative/antagonistic interactions can represent the two different, activating/inhibitive or trustful/mistrustful relationships which can often be found in real world systems [20].

Article [2] proposed a continuous-time model in which individuals may reach bipartite consensus, i.e., the agents split into two groups holding opposite values, instead of ordinary consensus. The fundamental digon sign-symmetry assumption in [2] can be removed, and relevant results have been obtained by [13]. As a matter of fact, continuous- or discrete-time Altafini-type protocols over static or time-varying signed networks have been widely investigated (see [20] for a review). But one should note that, unlike the two former classes of models which both have solid experimental or empirical evidence [9], [10], there has been few evidence-based research that focuses on the dynamics over signed networks.

Since interpersonal influences do not occur simultaneously and sequences of influences among people are complex [9], randomness also takes an important role in opinion dynamics. So no doubt random dynamical models over signed graphs have also been studied [17], [20], [21], [22], [23], and two different types of interactions along the negative links have been distinguished, i.e., the opposing negative dynamics [2], [21] and the repelling negative dynamics [22]. Intuitively, the former update rule makes agents take the opinions of negative neighbors into account, while the latter only makes the relative position of two individuals' opinions with negative ties farther away than before. [21] and [23] investigate models based on these two rules respectively and it is found that the two types of interaction rules can lead to extremely different behaviors.

In this paper, we will analyze an opinion formation model over signed networks based on opposing negative rules. For simplicity, we propose it in a gossip form, but from the proofs of Theorems, the update rule in this paper can be generalized and the extended model could be regarded as a description of a general heterogeneous random opinion formation process (see Remark 1 and note that the model in [21] is a homogeneous one). Besides, since the models

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in [17], [20], [21], [22], [23] are all linear, it may be possible to analyze the linear random models over signed networks from a random matrices perspective and this paper has made an attempt to connect models over signed networks with ordinary stochastic models which have been thoroughly studied. Thus, we can still obtain convergence conclusions using results on products of random matrices rather than sample path arguments, when the joint strongly connected assumption in [21] fails. It is also discovered that the slightly weakening of connectedness can lead to new behaviors for the random model. In this manner, we have a deeper understanding of dynamics over signed networks.

When a discussion takes place among a small group of people, we may assume that everyone can influence each other [9]. But when the group gets larger, a hierarchical structure, for instance, leaders and followers, could emerge. Thus we will introduce two different topological assumptions in this article, i.e., strongly connected and quasi-strongly connected. It is verified that the model behaviors are not the same under these two topologies.

The rest of this paper is organized as follows. In section II, we introduce some notations and preliminaries. The model and main results are presented in section III and simulations for illustration are shown in section IV. Section V discusses the conclusions and ideas for future work.

II. NOTATIONS AND PRELIMINARIES

Denote a simple directed graph (digraph) as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ represents the set of nodes (or agents) and $\mathcal{E}$ the set of arcs. With a slight abuse of notation, define a signed digraph as a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma)$, where $\sigma : \mathcal{E} \to \{+, -\}$ is a map that assigns each arc of $\mathcal{G}$ a sign. For some property P for digraphs, we say that a signed digraph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \sigma_0)$ has the property P if the digraph $(\mathcal{V}_0, \mathcal{E}_0)$ has that property.

An arc from node i to j is denoted by (i, j). A sequence of consecutive arcs $\{(i, k_1), (k_1, k_2) \cdots, (k_{l-1}, k_l), (k_l, j)\}$ is called a *path* from node i to j. We say that a node j is reachable from node i if there is a directed path from i to j. A digraph is referred to as *strongly connected* if each node is reachable from any other node. A signed graph $(\mathcal{V}', \mathcal{E}', \sigma')$ is a *subgraph* of $(\mathcal{V}, \mathcal{E}, \sigma)$ if $\mathcal{V}' \subset \mathcal{V}$, $\mathcal{E}' \subset \mathcal{E}'$ and $\sigma' = \sigma$ on $\mathcal{E}'$.

The concept of structurally balanced plays an important role in analyzing the dynamics over signed graphs [2]:

Definition 1: A signed digraph $\mathcal{G}$ is said to be structurally balanced if there exists a bipartition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of $\mathcal{V}$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $\sigma(i, j) = +, \forall i, j \in \mathcal{V}_l$ for $l \in \{1, 2\}$ and $\sigma(i, j) = -, \forall i \in \mathcal{V}_l, \forall j \in \mathcal{V} \setminus \mathcal{V}_l$ for $l \in \{1, 2\}$. It is called structurally unbalanced otherwise.

It is easy to know that an ordinary digraph can be viewed as a special case of a structurally balanced signed graph.

Given a square matrix $A \in \mathbb{R}^{n \times n}$, let A_{ij} be the (i, j)-th entry of A, and $\rho(A)$ the spectral radius of A. For two matrices $A, B \in \mathbb{R}^{m \times n}$, write $A \geq 0$ if $A_{ij} \geq 0$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$, and $A \geq B$ if $A - B \geq 0$. The relations $>$, $\leq$, $<$ are defined similarly.

A matrix M is said to be nonnegative if $M \geq 0$. A nonnegative matrix M is called stochastic (substochastic) if $M\mathbf{1} = (\leq)\mathbf{1}$ and a nonnegative vector α is declared to be a probability vector, if $\alpha^T \mathbf{1} = 1$, where $\mathbf{1}$ is the column vector of ones $(1, 1, \cdots, 1)^T$. For a matrix A, denote $|A| = [|A_{ij}|]$. If $|A|\mathbf{1} = \mathbf{1}$, then we say that A is an absolutely stochastic matrix. Denote the unit vector of corresponding size whose i-th component is 1 as e_i, the identity matrix of size n as I_n and $n \times m$ matrix with all entries zero as $\mathbf{0}_{nm}$.

A square matrix A is said to be adapted to a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma)$, if, for all nondiagonal entries, the sign of A_{ij} is the same as $\sigma(i, j)$ when $(i, j) \in \mathcal{E}$, and $A_{ij} = 0$ when $(i, j) \notin \mathcal{E}$. So for a nonnegative matrix A and an ordinary digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, A is adapted to $\mathcal{G}$ if and only if $A_{ij} > 0 \Leftrightarrow (i, j) \in \mathcal{E}$, for all $1 \leq i, j \leq n$, $i \neq j$. A stochastic matrix A is called irreducible if there exists a strongly connected ordinary digraph $\mathcal{G}$ to which A is adapted. For an absolutely stochastic matrix B, we say that B is irreducible if $|B|$ is irreducible. An absolutely stochastic matrix A is referred to as structurally (un)balanced if there exists a structurally (un)balanced signed graph $\mathcal{G}$ to which A is adapted.

A random matrix is a random variable taking values in the set of $n \times n$ matrices. For simplicity of notation, let $\{C(t), t \geq 0\}$ be a sequence of random matrices and l, k be some nonnegative integers, let

$$\Phi_C(k, l) = \begin{cases} C(k)C(k-1)\cdots C(l), & k \geq l, \\ I, & k < l, \end{cases} \tag{1}$$

and

$$\overleftarrow{\Phi}_C(l, k) = \begin{cases} C(l)C(l+1)\cdots C(k), & k \geq l, \\ I, & k < l. \end{cases} \tag{2}$$

III. MAIN RESULTS

A. Problem Formation

Consider a network with agents $\mathcal{V} = \{1, 2, \cdots, n\}$, $n \geq 3$. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma)$ be the interacting graph without self-loops, P be a nonnegative matrix adapted to $(\mathcal{V}, \mathcal{E})$ with $\mathbf{1}^T P \mathbf{1} = 1$, and W be a matrix adapted to $(\mathcal{V}, \mathcal{E}, \sigma)$ with $-\mathbf{1}\mathbf{1}^T \leq W \leq \mathbf{1}\mathbf{1}^T$.

For simplicity, we first introduce a discrete-time gossip model over signed networks as follows. Denote the belief vector at time t by $X(t)$, where the k-th component $X_k(t)$ is agent k's opinion at time t, $t \geq 0$ and $1 \leq k \leq n$. At each time slot, edge $(i, j) \in \mathcal{E}$ is selected with probability p_{ij}, independently of previous selections, and the agents' beliefs update as follows:

$$\begin{aligned} X_i(t+1) &= (1 - |w_{ij}|)X_i(t) + w_{ij}X_j(t) \\ X_k(t+1) &= X_k(t), \qquad \text{for } k \neq i. \end{aligned} \tag{3}$$

In other words, at time slot t, only agent i updates its opinion as a linear combination of agents i and j's previous beliefs while others do not update. If $\mathcal{G}$ has no negative arcs and hence is an traditional digraph, then the system becomes the classical gossip models with update rule (3) (in fact, it is an asymmetric gossip model as in [7]). Supposing that the term $(1-|w_{ij}|)$ in (3) becomes $(1-w_{ij})$, then (3) is the model in [22]. If $w_{ij}=\alpha>0$ for all $w_{ij}>0$ and $w_{ij}=\beta<0$ for all $w_{ij}<0$, then (3) serves as a special case of [21]. But since update rule (3) can be extended, the analysis of (3) is not trivial (see Remark 1).

The system (3) can also be rewritten in a compact form:

$$X(t+1)=W(t)X(t), \quad t\geq 0, \tag{4}$$

where $\{W(t), t\geq 0\}$ is a sequence of i.i.d. random matrices satisfying:

$$\mathbb{P}\{W(t)=I-|w_{ij}|e_ie_i^T+w_{ij}e_ie_j^T\}=p_{ij}, \tag{5}$$

where $\{W(t)\}$ are random absolutely stochastic matrices and $\mathbb{E}\{W(0)\}$ is an absolutely stochastic matrix with positive diagonal entries. The update rule (3) (or (4)) guarantees the existence of the evolution process and we denote the probability space capturing all the random components by $(\Omega,\mathcal{F},\mathbb{P})$.

Note that although random models in [21], [22], [23], [20] are all linear and (4) is the compact form of these models, $\{W(t)\}$ in these models may not be absolutely stochastic.

It is well known that the dynamics over signed networks may not reach consensus, but one can define modulus consensus [17]:

Definition 2: For system (3) with fixed initial value, if $\lim_{t\to\infty}|X_i(t)|=M$ a.s. for all $i\in\mathcal{V}$ and some nonnegative $M\in\mathbb{R}$, then we say that system (3) achieves modulus consensus. If M is a nonnegative random variable, then we say that (3) achieves probabilistic modulus consensus.

B. The Strongly Connected Case

We have the following conclusion if the underlying graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\sigma)$ is strongly connected.

Theorem 1: Suppose $\mathcal{G}=(\mathcal{V},\mathcal{E},\sigma)$ is strongly connected. If $W:=\mathbb{E}\{W(0)\}$ is structurally balanced, then for fixed initial value $X(0)$, system (3) converges and achieves probabilistic modulus consensus; if W is structurally unbalanced, then for fixed initial value $X(0)$, $\lim_{t\to\infty}X(t)=\mathbf{0}$ a.s.

The following three lemmas are needed to proof Theorem 1, and the proof of Lemma 3 is in the Appendix.

Lemma 1 (Lemma 4.2.5 in [25]): Let W be an irreducible absolutely stochastic matrix with positive diagonal entries. W is structurally balanced if and only if there exists a diagonal matrix D satisfying $D^2=I$ such that DWD is nonnegative and D is unique in the sense that if there exist two diagonal matrices D_1, D_2, satisfying $D_1^2=I$, $D_2^2=I$ such that D_1WD_1, $D_2WD_2\geq 0$, then $D_1=D_2$ or $D_1=-D_2$.

Lemma 2 (Corollary 4.2.1 in [25]): Let W be an irreducible absolutely stochastic matrix with positive diagonal entries. W is structurally unbalanced if and only if $\rho(W)<1$.

Lemma 3: Let $\{C(t), t\geq 0\}$ be a sequence of i.i.d. random matrices. If for $C:=\mathbb{E}\{C(t)\}$, $\rho(C)<1$, then $\lim_{t\to\infty}\Phi_C(t,0)=\mathbf{0}$ a.s., where $\Phi_C(t,0)$ is defined in (1).

Proof of Theorem 1: Since W is structurally balanced and irreducible with positive diagonal entries, by Lemma 1, there exists a diagonal matrix D, with $D^2=I$, such that $A=DWD$ is nonnegative. Hence A is an irreducible aperiodic stochastic matrix. From Perron's theorem in [11], 1 is the largest eigenvalue of A in absolute value with multiplicity 1. Let T be a nonsingular matrix leading A to the Jordan canonical form $TAT^{-1}=\begin{bmatrix}\Lambda & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}$, where Λ is an $n-1$ dimensional matrix with all eigenvalues less than 1 in absolute value. Since A is aperiodic, irreducible and stochastic, there exists a unique probability vector α such that $\alpha^TA=\alpha^T$ from matrix theory. As a result, the last row in T must be $c\alpha^T$ with c being a constant. Without loss of generality, setting $c=1$ we have $T=\begin{bmatrix}\tilde{T}\\ \alpha^T\end{bmatrix}$ and, similarly, $T^{-1}=\begin{bmatrix}\tilde{U} & \mathbf{1}\end{bmatrix}$. Moreover, $\tilde{T}\mathbf{1}=\mathbf{0}$ and $\alpha^T\tilde{T}=\mathbf{0}$ since $TT^{-1}=I$. Define $A(t):=DW(t)D$ and thus $A(t)$ is a stochastic matrix.

Let $\tilde{X}(t)=TDX(t)$ for $t\geq 0$, then

$$\begin{aligned}\tilde{X}(t+1)&=TA(t)T^{-1}\tilde{X}(t)\\ &=\begin{bmatrix}\tilde{T}A(t)\tilde{U} & \tilde{T}A(t)\mathbf{1}\\ \alpha^TA(t)\tilde{U} & \alpha^TA(t)\mathbf{1}\end{bmatrix}\tilde{X}(t)\\ &=\begin{bmatrix}\tilde{T}A(t)\tilde{U} & \mathbf{0}\\ \alpha^TA(t)\tilde{U} & 1\end{bmatrix}\tilde{X}(t).\end{aligned}$$

The last equation holds since $A(t)$ is a stochastic matrix and $\tilde{T}\mathbf{1}=\mathbf{0}$.

Now denote the first $n-1$ components of $\tilde{X}(t)$ by $Y(t)$, the n-th component of $\tilde{X}(t)$ by $\tilde{X}(t)_n$ and $\tilde{T}A(t)\tilde{U}$ by $\bar{A}(t)$. We have

$$Y(t)=\Phi_{\bar{A}}(t,0)Y(0)$$

and

$$\tilde{X}(t)_n=\alpha^TX(0)+\alpha^T\sum_{i=1}^{t}A(i)\tilde{U}\Phi_{\bar{A}}(i-1,0)Y(0)$$

Since $\rho(\Lambda)<1$, by Lemma 3, $\lim_{t\to\infty}\Phi_{\bar{A}}(t,0)=\mathbf{0}$ a.s., and consequently, $\lim_{t\to\infty}Y(t)=0$ a.s. Because of the independence of $\{A(t)\}_{t\geq 0}$, same as the proof of Lemma 3, it follows that $\tilde{X}(t)_n$ converges a.s. to a random variable $Z=\alpha^TX(0)+\alpha^T\sum_{i=1}^{\infty}A(i)\tilde{U}\Phi_{\bar{A}}(i-1,1)Y(0)$. Therefore $\lim_{t\to\infty}\tilde{X}(t)=(0\ \cdots\ 0\ Z)^T$ and $\lim_{t\to\infty}DX(t)=\lim_{t\to\infty}T^{-1}\tilde{X}(t)=Z\mathbf{1}$ a.s., which implies that (4) converges a.s. and achieves probabilistic modulus consensus.

When W is irreducible and structurally unbalanced, note that W has positive diagonal entries. So the conclusion follows from Lemmas 2 and 3. ■

Remark 1: Theorem 1 is consistent with the classical convergence result of dynamics over signed networks [2] and of the random counterpart [21]. It should be noted that the proof of Theorem 1 only requires that $\{W(t)\}$ are independent and have the same expectation. Thus different gossip models, such as symmetric gossip, synchronous asymmetric gossip, broadcasting models and so on [7], subject to the opposing negative rule would also reach modulus consensus as long as the former condition holds. Moreover, $\{W(t)\}$ could take values in the set of absolutely stochastic matrices which are adapted to the subgraphs of $\mathcal{G}$, that is,

$$\mathbb{P}\{W(t) = B_k\} = p_k, \tag{6}$$

where $1 \le k \le m$ for some m. B_k is absolutely stochastic matrices with nonnegative diagonal entries and adapted to some subgraph of $\mathcal{G}$, and $p_k > 0$ with $\sum p_k = 1$. The conclusion still holds as long as $\mathbb{E}\{W(t)\}$ is irreducible and has positive diagonal entries. The intuitive meaning of the extended model is that, for a crowd of people who can discuss with each other in small groups, they will reach modulus (or bipartite) consensus as long as they can receive opinions of anyone else in the crowd. Of course these heterogeneous update rules no longer serve as special cases of the homogeneous model in [21].

C. Quasi-Strongly Connected Case

Quasi-strongly connected (a digraph with a directed spanning tree) is also a common topology assumption [13], [15], and under this assumption we have some conclusions similar to Theorem 1. Let us first introduce a few definitions and notations.

If a digraph $(\mathcal{V}, \mathcal{E})$ has paths from some node i to every other node j, then $(\mathcal{V}, \mathcal{E})$ is said to have a directed spanning tree including i as its root. For a signed digraph $\mathcal{G}$ with a spanning tree and m roots, we always sort $\mathcal{V}$ such that $\mathcal{V}_r = \{n-m+1, \dots, n\}$ is the node set of the roots. The subgraph $\mathcal{G}_r = (\mathcal{V}_r, \mathcal{E}|_{\mathcal{V}_r}, \sigma|_{\mathcal{V}_r})$ is called the rooted graph of $\mathcal{G}$. As a result, if a square matrix W is adapted to a signed digraph $\mathcal{G}$ with a spanning tree, then it always has the following form:

$$W = \begin{bmatrix} W_{11} & W_{12} \\ \mathbf{0} & W_r \end{bmatrix}, \tag{7}$$

where W_r is adapted to the rooted graph $\mathcal{G}_r$.

When the rooted graph is structurally unbalanced, the system converges to zero a.s.

Theorem 2: Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma)$ has a spanning tree. If $W_r := \mathbb{E}\{W_r(0)\}$ is structurally unbalanced, then for fixed initial value $X(0)$, $\lim_{t\to\infty} X(t) = \mathbf{0}$ a.s., where W_r is adapted to the rooted graph of $\mathcal{G}$.

Proof: Since $W = \mathbb{E}\{W(0)\}$, the diagonal entries of W are positive. Note that W has the form $W = \begin{bmatrix} W_{11} & W_{12} \\ \mathbf{0} & W_r \end{bmatrix}$. Thus Lemma 2 guarantees $\rho(W_r) < 1$ and Lemma 4 ensures $\rho(W_{11}) < 1$, which implies $\rho(W) < 1$. The conclusion follows from Lemma 3. ∎

New behaviors of the model may emerge when W_r is structurally balanced:

Proposition 1: Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma)$ has a spanning tree.
(i) If $W := \mathbb{E}\{W\}$ is structurally balanced, then for fixed initial value $X(0)$, system (3) converges and achieves probabilistic modulus consensus;
(ii) If $W_r := \mathbb{E}\{W_r(0)\}$ is structurally balanced, then for fixed initial value $X(0)$, the opinions of agents in the rooted graph converge and achieve probabilistic modulus consensus. Denote these two limit random variables as M and $-M$ and without loss of generality suppose $M \ge 0$. The other agents' beliefs will finally lie in $[-M, M]$. Moreover, if an agent i is influenced by both partition of the roots, then $\limsup_{t\to\infty} X_i(t) = -\liminf_{t\to\infty} X_i(t) = M$ a.s.

Proof: When W is structurally balanced, by Lemma 2.3 in [13], there exists a diagonal matrix D, with $D^2 = I$, such that $A = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_r \end{bmatrix}$ is nonnegative. From the definition of a digraph root, we know that at least one of the row sum of A_{11} is less than one. So it follows that $\rho(A_{11}) < 1$ by Lemma 4 in the Appendix. Because A_r is irreducible and aperiodic, 1 is the largest eigenvalue of A in absolute value with multiplicity 1. Let $T = \begin{bmatrix} T_1 & T_2 \\ \mathbf{0} & \alpha^T \end{bmatrix}$ and the rest of the proof is similar to that of Theorem 1, where T is a nonsingular matrix leading A to the Jordan canonical form $TAT^{-1} = \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$ and α is the unique invariant measure of A_r satisfying $\alpha^T A_r = \alpha^T$.

When W_r is structurally balanced, from Theorem 1 it follows that the opinions of agents in the rooted graph converge and achieve probabilistic modulus consensus. Thus for fixed $\varepsilon > 0$, define $S_1 := \inf\{t \ge 0 : ||X_k(t)| - M| < \varepsilon, \forall k \in \mathcal{V}_r\}$ and S_1 is a finite stopping time.

Since system (3) is bounded and $\mathcal{G}$ has a spanning tree, there exists a fixed integer $K(\varepsilon)$ and positive constants p_* such that $||X_i(S_1 + K)| - M| < 2\varepsilon$ for all $i \in \mathcal{V} \setminus \mathcal{V}_r$ with positive probability p_*. This is because the model (3) is an asymmetric gossip model, and we could first choose the arcs from roots to their neighbors repeatedly after time S_1 such that the states of roots' neighbors are close enough to M (this can be done even when $|w_{ij}| = 1$). Then use the same selection method recursively. Denote this selection process as $\mathcal{W}(S_1, S_1 + K)$ (actually, it is a product of selected matrices) and define $\mathcal{W}(S_1 + qK, S_1 + (q+1)K)$, $q \ge 1$, in the same way. Note that $\{W(t)\}$ are i.i.d. and S_1 is a finite stopping time, so $\mathcal{W}(S_1 + qK, S_1 + (q+1)K)$, $q \ge 0$, are independent and have positive probability (Theorem 4.1.3 in [5]). By Borel-Cantelli Lemma, we have $\mathbb{P}\{\mathcal{W}(S_1 + qK, S_1 + (q+1)K) \text{ i.o.}\} = 1$.

Because of the linearity of model (3), $||X_k(t)| - M| < \varepsilon$, $\forall k \in \mathcal{V}_r$ and $t \ge S_1$. Therefore,

$$\begin{aligned} &\{\limsup_{t\to\infty} |X_i(t)| \le M + 2\varepsilon, \forall i \in \mathcal{V}\} \\ &\supseteq \{|X_i(T)| \le M + 2\varepsilon, \forall i \in \mathcal{V} \text{ and some } T(\omega)\} \\ &\supseteq \{[||X_k(t)| - M| < \varepsilon, \forall k \in \mathcal{V}_r, t \ge S_1] \\ &\qquad \cap [\mathcal{W}(S_1 + qK, S_1 + (q+1)K) \text{ i.o.}]\} \end{aligned}$$

The second assertion of (ii) follows from that the last set has probability 1 and the last assertion of (ii) can be obtained in the same way. ∎

Under the condition of Proposition 1 (ii), the opinions of the other agents may not converge, but the expectations of the opinions converge and will finally lie in the interval $[-\mathbb{E}\{M\}, \mathbb{E}\{M\}]$ by Theorem 2 in [26]. The conclusions obtained in this section is quite similar to that in [26]: For

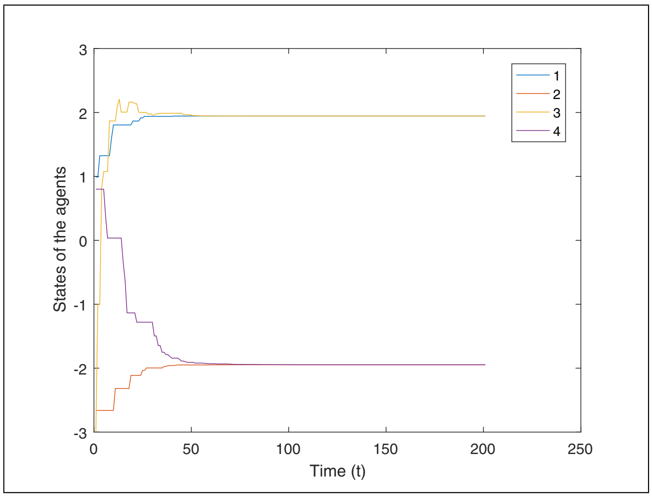

Fig. 1. A sample path of (3) over signed network Fig. 4 (a). The system reaches probabilistic bipartite consensus.

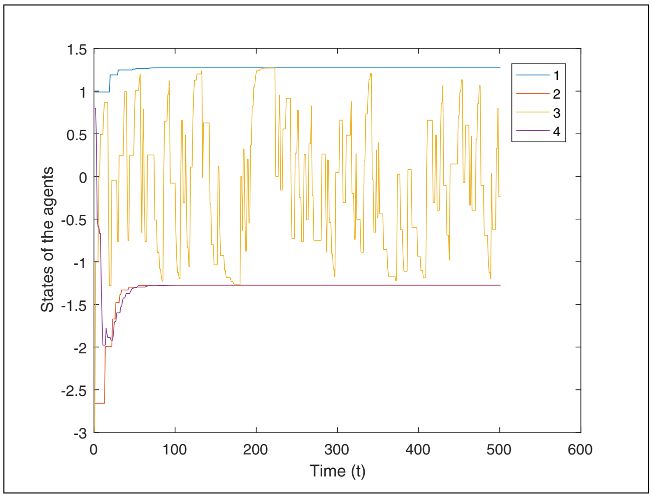

Fig. 2. A sample path of (3) over signed network Fig. 4 (b). The opinions of the roots, i.e., agents 1 and 2 converge. Note that the belief of agent 3 ends in fluctuating but lies between agents 1 and 2.

deterministic models, if the underlying graph is static and has a spanning tree, then the rooted graph reach bipartite consensus and the opinions of the other agents converge and are convex combinations of the roots' opinions [14]; however, if the underlying graph is time-varying, the beliefs of the other agents may not converge but be bounded by the limit of roots' opinions. Also note that, the convergence results in this section can also be extended to the general model in Remark 1.

IV. Simulations

This section is an illustration of section III-C. Consider a network with 4 agents and different interacting graphs shown in Fig. 4, one sample path of each system is shown in Fig. 1-3 respectively.

V. Conclusion

In this paper, we analyze a gossip model based on opposing negative rules as a partial counterpart of the gossip

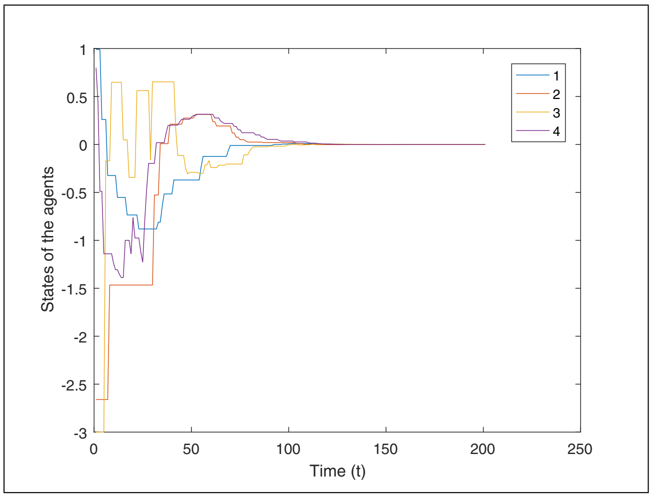

Fig. 3. A sample path of (3) over signed network Fig. 4 (c). The opinions of all agents converge to zero.

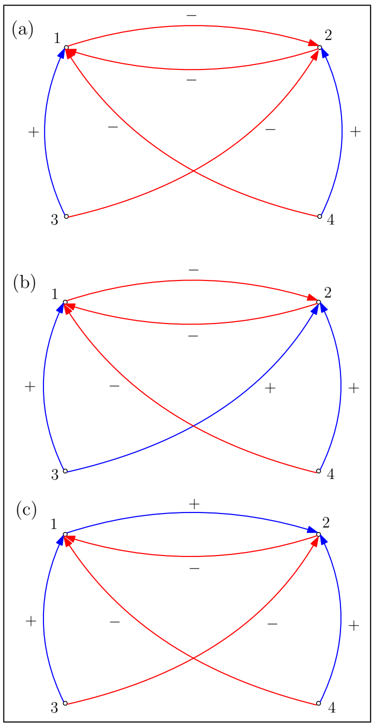

Fig. 4. A signed network with 4 agents. Agents 1 and 2 are roots. For (a), the network is structurally balanced; for (b), only the rooted graph is structurally balanced; for (c), the rooted graph is structurally unbalanced.

model based on repelling negative dynamics [22]. Of course, the original random selection process of this article's model serves as a special case of the random graph process of [21] under certain circumstances, but the former can be extended to a more general heterogeneous model while the conclusions still hold. This article can be regarded as an example of applying results on products of random stochastic matrices to analyze the dynamics over signed networks, which makes proofs much simpler ([17]'s efforts focused on deterministic models). Further work would be to utilize more general results on products of random stochastic matrices (e.g. [24])

to release the independent assumption, and use conclusions on products of random matrices (since the random matrices in [21], [23] may not be absolutely stochastic) to solve the remaining threshold problems in [21]. Since the opinion formation in the real world may not take place over a strongly or quasi-strongly connected network, other future work would be to explore the behaviors of random models over more general topologies, for example, networks with stubborn agents or strongly connected components.

APPENDIX

Proof of Lemma 3: Since $\rho_0 = \rho(C) < 1$, from the Jordan canonical decomposition, we have for all $k \geq 1$, $\|C^k\|_\infty \leq qk^{n-1}\rho_0^k$, where $\|\cdot\|_\infty$ is the maximum row sum matrix norm and q is a constant depending on C only. Note that $\mathbb{E}\{C(t)\} = C$, thus $\mathbb{E}\{\|\Phi_C(k-1,0)\|_1\} \leq n\|\mathbb{E}\{\Phi_C(k-1,0)\}\|_\infty \leq qnk^{n-1}\rho_0^k$, where $\|\cdot\|_1$ is the maximum column sum matrix norm. Fix $v \in (\rho_0, 1)$, from Chebychev inequality, it follows that for all $k \geq 1$,

$$\mathbb{P}\{\|\Phi_C(k-1,0)\|_1 \geq v^k\} \leq \beta_k,$$

where $\beta_k = qnk^{n-1}(\rho_0/v)^k$. Since $\sum \beta_k < \infty$, by Borel-Cantelli Lemma $\|\Phi_C(k-1,0)\|_1 < v^k$ for all but finitely many k with probability one, which implies that $\lim_{t\to\infty} \Phi_C(t,0) = \mathbf{0}$ a.s. ■

Lemma 4 (Lemma 4 in [8]): Consider a substochastic matrix $M \in \mathbb{R}^{n\times n}$. If for every i, $1 \leq i \leq n$, there exists an integer j, $1 \leq j \leq n$, with the sum of j-th row less than 1 and a sequence of distinct integers $k_1 = i,\ k_2, \dots,\ k_m = j,\ 1 \leq m \leq n$, such that $m_{k_1k_2}m_{k_2k_3}\cdots m_{k_{m-1}k_m} > 0$, then $\rho(M) < 1$.

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